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Franz X. Hof
Vienna University of Technology

Klaus Prettner
University of Hohenheim

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Relative consumption, relative wealth, and long-run growth: When and why is the standard analysis prone to erroneous conclusions?

Franz X. Hof[a] and Klaus Prettner[b*]

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a) Vienna University of Technology
Institute of Statistics and Mathematical Methods in Economics
Research Unit Economics (105-3)
Wiedner Hauptstr. 8 – 10
1040 Vienna, Austria
email: franz.hof@tuwien.ac.at

b) University of Hohenheim
Institute of Economics
Schloss, Osthof-West
70593 Stuttgart, Germany
email: klaus.prettner@uni-hohenheim.de
Corresponding Author

Abstract

We employ a novel approach for analyzing the effects of relative consumption and relative wealth preferences on both the decentralized and the socially optimal economic growth rates. In the pertinent literature these effects are usually assessed by examining the dependence of the growth rates on the two parameters of the instantaneous utility function that *seem* to measure the strength of the relative consumption and the relative wealth motive. We go beyond the sole consideration of parameters by revealing the fundamental factors that *ultimately* determine long-run growth. In doing so we identify widely used types of status preferences in which the traditional approach is prone to erroneous conclusions. For example, in one of these specifications the parameter that *seems* to determine the strength of the relative consumption motive actually also affects the strength of the relative wealth motive and the elasticity of intertemporal substitution.

JEL classification: D31, D62, O10, O30.
Keywords: Relative consumption, relative wealth, quest for status, long-run economic growth, social optimality, deep factors.

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1 Introduction

We propose a novel approach to reexamine the implications of both relative consumption and relative wealth preferences.[1] The analysis is carried out in the context of an otherwise standard AK-model of endogenous growth with homogeneous agents and exogenous labor supply. In the pertinent literature it is common practice to analyze the implications of such preferences in the following way: First, a functional form of the instantaneous utility function is chosen that has two crucial properties: i) it allows for the existence of a balanced growth path (BGP), and ii) for mathematical convenience it contains as few parameters as possible. Second, the effects of relative consumption and relative wealth preferences are assessed by analyzing the dependence of the BGP growth rate on the two parameters of the instantaneous utility function that *seem* to be the appropriate measures of the strength of the relative consumption and the relative wealth motive. The aim of this paper is to show that this standard method of analysis involves a great risk of drawing erroneous conclusions.

In order to give both mathematical and economic explanations for the potential fallacies we go beyond the consideration of parameters by putting special emphasis on the identification of the *fundamental factors* that ultimately determine both the decentralized and the socially optimal long-run growth rates. These fundamental factors are connected to technology and preferences. In our analysis we focus on the three fundamental factors that are linked to the specification of the instantaneous utility function because they are appropriate measures of the household's willingness to substitute i) relative consumption for absolute consumption, ii) relative wealth for absolute consumption, and iii) future absolute consumption for current absolute consumption.

In our approach it becomes possible to analyze the effects of *ceteris paribus changes* in i) the strength of the relative consumption motive, ii) the strength of the relative wealth motive, and iii) the magnitude of the effective elasticity of intertemporal substitution by considering a change in the corresponding fundamental factor holding all else equal. In the standard approach, however, such thought experiments cannot be carried out if specifications of utility functions are used in which the crucial parameters affect more than one out of these three fundamental factors. In such instances the standard approach is prone to erroneous conclusions. For example, in a standard specification that we analyze in detail below, a certain parameter seems to affect only the strength of the relative consumption motive. Our fundamental factor approach, however, shows that this parameter also influences both the strength of the relative wealth motive and the willingness to substitute absolute consumption intertemporally. The standard approach is unaware of the latter two effects due to its ignorance of the fundamental factors. Hence, it does not see any necessity to decompose the total reaction of the growth rate that results from

[1] In the literature, it is common practice to focus either on relative consumption or on relative wealth. For specifications that employ relative consumption (or more general consumption externalities) see, for instance, Abel (1990, 2005), Galí (1994), Harbaugh (1996), Carroll et al. (1997), Rauscher (1997), Grossmann (1998), Fisher and Hof (2000), Ljungqvist and Uhlig (2000), Liu and Turnovsky (2005), Turnovsky and Monteiro (2007), Fisher and Heijdra (2009), Barnett et al. (2010), and Strulik (2015). Examples of the relative wealth approach are Corneo and Jeanne (1997, 2001a,b), Futagami and Shibata (1998), Fisher and Hof (2005, 2008), Van Long and Shimomura (2004), García-Peñalosa and Turnovsky (2008), and Fisher (2010). For frameworks that allow for both specifications see Tournemaine and Tsoukis (2008), Riegler (2009), Ghosh and Wendner (2014), Ghosh and Wendner (2018), Wendner (2015), and Klarl (2017).

a change in the parameter into the three effects that correspond to the changes in the three fundamental factors. Instead, it attributes the total growth effect erroneously to the change in the strength of the relative consumption motive. It is thus possible that the resulting assertions of the standard approach with respect to the implications of relative consumption preferences on long-run growth are not only quantitatively, but also qualitatively flawed.

With our approach, we reexamine specifications of the instantaneous utility function that are used in the pertinent literature. Three of them deal with the *pure* relative consumption approach that abstracts from the relative wealth motive, whereas the three other specifications consider the general case in which both relative consumption and relative wealth matter for utility. In the context of the pure relative consumption approach we provide two prominent specifications of preferences in which the standard approach leads to erroneous conclusions. Their common feature is that the instantaneous utility function is obtained by applying an isoelastic (CRRA-type) transformation to a *geometric weighted average* of absolute consumption and relative consumption. The standard approach employs the derivative of the BGP growth rate with respect to the weight of relative consumption to assess the implications of relative consumption preferences. In so doing it fails to notice that a rise in the weight of relative consumption is inevitably associated with a decrease in the weight of absolute consumption which, in turn, changes the effective elasticity of intertemporal substitution provided that the parameter of the CRRA-type transformation is unequal to unity. In other words, the standard approach is unaware of the fact that, in general, changes in the weight of relative consumption also lead to changes in the willingness to substitute absolute consumption intertemporally and, hence, must not be interpreted as ceteris paribus changes in the strength of the relative consumption motive. In our third illustration of the pure relative consumption case the standard approach yields correct results because the corresponding utility function exhibits no functional dependence between its parameters.

The analysis becomes more complicated if the concept of the geometric weighted average is applied to situations in which also relative wealth matters. Due to the fact that the sum of the three weights equals unity by definition, a change in the weight of relative consumption affects not only the strength of the relative consumption motive, but also the effective elasticity of intertemporal substitution and/or the strength of the relative wealth motive. We illustrate the resulting problems by means of two standard specifications. However, we also analyze a seminal case in which the standard approach yields correct results and explain why this is the case.

The problems of the standard approach that result from the presence of a functional dependence between the parameters of the instantaneous utility function apply also to the analysis of the socially optimal growth rate. Flawed conclusions might be drawn if parameters exist that at first glance seem to determine only the strength of the relative consumption or the relative wealth motive, but actually also affect the willingness to substitute absolute consumption intertemporally. More precisely, our fundamental factor approach shows that the socially optimal growth rate is independent of both relative consumption and relative wealth preferences. This, in turn, implies that any effect of such preferences on the socially optimal growth rate that is detected by the standard analysis results exclusively from the unintended and unnoticed effect of parameter changes in the instantaneous utility function on the effective elasticity of intertemporal substitution.

In providing appropriate economic interpretations of the effects of relative wealth and relative consumption preferences and explaining the pitfalls of the standard analysis we draw heavily on the Euler equation that governs the dynamic evolution of aggregate consumption in a symmetric macroeconomic equilibrium. This Euler equation differs from its counterpart in the standard model in the following respects: 1) If relative wealth matters for utility, then the market rate of return is replaced by the *effective* rate of return. The latter is defined as the sum of the market rate and an extra return that results from social comparisons that are based on both relative wealth and relative consumption. More precisely, the comparison-induced extra return is the marginal rate of substitution (MRS) of absolute wealth for absolute consumption. A very helpful technical trick of the paper is to express the comparison-induced extra return as the product of the consumption-wealth ratio and the comparison-induced extra return *factor* that has the following properties: i) It depends positively on the strength of the relative wealth motive, irrespective of the strength of the relative consumption motive. This property ensures that the willingness to save always depends positively on the strength of the relative wealth motive. ii) If relative wealth matters for utility, then it depends negatively on the strength of the relative consumption motive. The implied decrease in the willingness to save results from the fact that any decrease in absolute consumption is associated with a reduction in relative consumption that leads to an additional reduction in instantaneous utility. iii) In the absence of the relative wealth motive it is identical to zero so that the comparison-induced extra return vanishes irrespective of the strength of the relative consumption motive. Consequently, the effective rate of return equals the market rate. 2) In principle, the *effective* elasticity of intertemporal substitution may depend on the strength of the relative consumption motive. This possibility vanishes, however, when we introduce weak restrictions on the utility function that are sufficient for the existence of a balanced growth path (BGP).

The properties given in 1) and 2) imply that relative consumption and relative wealth preferences influence the BGP growth rate directly only via their effect on the comparison-induced extra return but not via the effective elasticity of intertemporal substitution. There is no indirect effect via the market rate of return that equals the marginal product of capital as perceived by the representative firm. This is due to the fact that we restrict our attention to the case in which labor supply is exogenously given.

The paper is organized as follows. In Section 2 we describe the assumptions of the model and study the optimal behavior of households and firms. In Section 3 we consider the macroeconomic equilibrium of the decentralized economy. We derive conditions for the existence of a BGP and analyze the long-run effects of relative consumption and relative wealth preferences by means of the corresponding fundamental factors. In Section 4 we discuss widely-used specifications of the instantaneous utility function in which the ignorance of the fundamental factors most likely leads to erroneous conclusions. In Section 5 we analyze the difference between the decentralized BGP and its socially optimal counterpart. Once again we show the pitfalls of ignoring the fundamental factors. Finally, in Section 6 we conclude and outline the scope of further research.

2 The model

2.1 Households

Consider a continuum of infinitely-lived identical households with mass 1. The flow budget constraint of the representative household is given by

$$\dot{a} = ra + wl - c, \tag{1}$$

where a refers to net assets, r denotes the real rental rate of physical capital, which is equal to the real interest rate because we abstract from depreciation, w is the real wage, l refers to hours worked, and c denotes consumption.

Instantaneous utility depends not only on absolute consumption c, but also on relative consumption c/C and/or on relative wealth a/A, where C denotes average consumption, while A is average wealth. We restrict our attention to the case in which labor supply is exogenously given so that the appropriate general specification of the instantaneous utility function takes the form $u = u(c, c/C, a/A)$. The representative household derives positive and diminishing marginal utility from absolute consumption and nonnegative marginal utility from both relative consumption and relative wealth:

$$u_c > 0, \quad u_{cc} < 0, \quad u_{c/C} \geq 0, \quad u_{a/A} \geq 0, \quad u_{c/C} > 0 \vee u_{a/A} > 0. \tag{2}$$

The last assumption given in (2) rules out the uninteresting specification in which neither relative consumption nor relative wealth matter.[2] It will prove helpful to use the fact that instantaneous utility can be expressed as

$$V = V(c, C, a, A) \equiv u(c, c/C, a/A). \tag{3}$$

To ensure a well-behaved intertemporal optimization problem, we assume that the function $V(c, C, a, A)$ is i) strictly concave in c, and ii) jointly strictly concave in c and a in case relative wealth matters for utility:

$$V_{cc} < 0, \qquad \text{and} \qquad V_{cc}V_{aa} - (V_{ca})^2 > 0 \quad \text{if} \quad u_{a/A} > 0. \tag{4}$$

The expressions for V_{cc} and $V_{cc}V_{aa} - (V_{ca})^2$ are given in Appendix A.1.

The representative household maximizes overall utility as given by $\int_0^\infty e^{-\rho t} u(c, c/C, a/A)\, dt$, where ρ denotes the discount rate, subject to the flow budget constraint (1) and the initial condition $a(0) = a_0$ by choosing the time path of absolute consumption c. An important aspect of this optimization problem is that the representative household takes not only the time paths of the real wage w and the real interest rate r, but also the time paths of average consumption

[2] The most common interpretation of relative consumption and relative wealth preferences is based on status preferences in the sense that $u(c, c/C, a/A) \equiv \tilde{u}(c, s(c/C, a/A))$. In this specification, instantaneous utility depends positively on both absolute consumption c and status s, $\tilde{u}_c > 0$ and $\tilde{u}_s > 0$, while status depends nonnegatively on relative consumption and relative wealth, $s_{c/C} \geq 0$ and $s_{a/A} \geq 0$, where, in addition, $s_{c/C} > 0 \vee s_{a/A} > 0$ holds. Obviously, the general specification $u = u(c, c/C, a/A)$ encompasses the status interpretation but does not rule out alternative explanations.

C and average wealth A as given. The current-value Hamiltonian of the optimization problem is given by $H = u(c, c/C, a/A) + \lambda(ra + wl - c)$, where the costate variable λ denotes the shadow price of absolute wealth. The necessary optimality conditions for an interior equilibrium, $H_c = 0$ and $\dot{\lambda} = \rho\lambda - H_a$, can be written as

$$\lambda = u_c(c, c/C, a/A) + u_{c/C}(c, c/C, a/A)\, C^{-1}, \tag{5}$$

$$\dot{\lambda} = -\left[r\lambda + u_{a/A}(c, c/C, a/A)\, A^{-1} - \rho\lambda\right]. \tag{6}$$

The assumptions given in (4) ensure that if the transversality condition

$$\lim_{t\to\infty} e^{-\rho t}\lambda a = 0 \tag{7}$$

is satisfied, then the necessary optimality conditions (5) and (6) are also sufficient.

From the first-order conditions it follows that the growth rate of the costate variable λ is

$$\frac{\dot{\lambda}}{\lambda} = -\left[r + \frac{u_{a/A}(c, c/C, a/A)\, A^{-1}}{u_c(c, c/C, a/A) + u_{c/C}(c, c/C, a/A)\, C^{-1}} - \rho\right]. \tag{8}$$

The sum of the first two terms within square brackets is the *effective* rate of return, where the first term, r, is the standard market rate of return, while the second term measures the extra return that results from social comparisons based on relative wealth and relative consumption. For the sake of brevity, we henceforth use the compact notion *comparison-induced extra return.*

To provide a thorough economic interpretation of the comparison-induced extra return, we first employ (3) to rewrite the first-order conditions (5) and (6) as $\lambda = V_c(c, C, a, A)$ and $\dot{\lambda} = -[\lambda r + V_a(c, C, a, A) - \rho\lambda]$. This, in turn, implies that (8) can also be expressed as

$$\frac{\dot{\lambda}}{\lambda} = -\left[r + \frac{V_a(c, C, a, A)}{V_c(c, C, a, A)} - \rho\right]. \tag{9}$$

From

$$-\frac{u_{a/A}A^{-1}}{u_c + u_{c/C}C^{-1}} = -\frac{V_a}{V_c} = \left.\frac{dc}{da}\right|_{dV=0, dC=dA=0}$$

it is then obvious that the comparison-induced extra return is the marginal rate of substitution (MRS) of absolute wealth a for absolute consumption c. Hence, it tells us the amount of absolute consumption c that the consumer – who takes both average consumption C and average wealth A as given – would be willing to give up for a one-unit marginal increase in absolute wealth a. The derivation and interpretation of this MRS are straightforward: On the one hand, for a given value of A, an increase in absolute wealth $da > 0$ leads to a rise in relative wealth a/A as given by $d(a/A) = A^{-1}da > 0$. As long as relative wealth matters for utility so that $u_{a/A} > 0$, the resulting increase in instantaneous utility is given by $V_a da = u_{a/A}A^{-1}da > 0$. On the other hand, for a given value of average consumption C, a decrease in absolute consumption $dc < 0$ is accompanied by a fall in relative consumption c/C as given by $d(c/C) = C^{-1}dc < 0$. If relative consumption matters for utility so that $u_{c/C} > 0$, then not only the fall in absolute consumption, but also the decline in relative consumption causes instantaneous utility to decrease with the total effect being $V_c dc = \left[u_c + u_{c/C}C^{-1}\right] dc < 0$. Obviously, instantaneous utility remains unchanged if and only if the change in absolute consumption satisfies the condition that $dc = -(V_a/V_c)\,da = -u_{a/A}A^{-1}\left[u_c + u_{c/C}C^{-1}\right]^{-1} da$. Please note that the presence of relative consumption preferences exerts ceteris paribus a negative effect on the magnitude of the MRS of a for c. In other words, the consumer is willing to forgo a smaller amount of absolute consumption because the fall in c is associated with a decrease in relative consumption c/C where the latter effect leads to an additional reduction in instantaneous utility.

It is decisive for the rest of the paper to rewrite (8) as

$$\frac{\dot{\lambda}}{\lambda} = -\left[r + \frac{m^{a/A}\left(c, c/C, a/A\right)}{1 + m^{c/C}\left(c, c/C, a/A\right)} \times \frac{c}{a} - \rho\right], \tag{10}$$

where

$$m^x = m^x\left(c, c/C, a/A\right) \equiv \frac{x}{c} \times \frac{u_x\left(c, c/C, a/A\right)}{u_c\left(c, c/C, a/A\right)}, \qquad \text{for} \qquad x = \frac{c}{C}, \frac{a}{A}. \tag{11}$$

Obviously, u_x/u_c is the *standard* marginal rate of substitution (MRS) of x for absolute consumption c, where x is either relative consumption or relative wealth. From

$$\left.\frac{dc}{dx}\right|_{du=0} = -\frac{u_x}{u_c} \Rightarrow \left.\frac{dc/c}{dx/x}\right|_{du=0} = -\frac{x}{c} \times \frac{u_x}{u_c} = -m^x, \qquad \text{for} \qquad x = \frac{c}{C}, \frac{a}{A}$$

it follows that $m^x \geq 0$ is the *percentage-MRS* of x for c. More precisely, $m^{c/C}$ refers to the *percent* of absolute consumption c the consumer would be willing to forgo to raise relative consumption c/C by one *percent* for a given value of relative wealth a/A. Analogously, the term $m^{a/A}$ refers to the *percent* of absolute consumption c the household would be willing to sacrifice to raise relative wealth a/A by one *percent* for a given value of relative consumption c/C. The assumptions that we introduce below to ensure the existence of a BGP imply that the *percentage*-MRS (m^x) – in contrast to the *standard* MRS (u_x/u_c) – is constant along the BGP. Due to this advantage we henceforth employ the percentage-MRS instead of its standard counterpart.

Obviously, $m^{c/C}$ and $m^{a/A}$ are appropriate measures of the strength of the relative consumption motive and the relative wealth motive, respectively. In addition, if relative consumption matters for utility, $m^{c/C} > 0$, then $0 \leq m^{a/A}/m^{c/C} < \infty$. This ratio represents the percentage-MRS of relative wealth a/A for relative consumption c/C. It gives the *percent* of relative consumption c/C the consumer would be willing to sacrifice to raise relative wealth a/A by one *percent* for a given value of absolute consumption c.

Remark 1. *An alternative economic interpretation of the terms between brackets on the right-hand side of (10) can be found in Tournemaine and Tsoukis (2008). These authors follow Futagami and Shibata (1998) and dub an expression that is analogous to*

$$\rho - \frac{m^{a/A}\left(c, c/C, a/A\right)}{1 + m^{c/C}\left(c, c/C, a/A\right)} \times \frac{c}{a},$$

(but does not employ the concept of the percentage-MRS of a/A or c/C for c) as the effective discount rate. On page 316 they state: "... seeking greater social status in wealth makes people more patient, whereas seeking greater social status in consumption makes them more impatient."

In this paper we prefer to employ the concept of the effective rate of return that also occurs in the transversality condition (13) given below.

The introduction of $m^{c/C}$ and $m^{a/A}$ has further implications for the representation of equations and conditions. For instance, the first-order condition (5) with respect to the optimal choice of absolute consumption c can be rewritten as

$$\lambda = u_c\left(c, c/C, a/A\right)\left[1 + m^{c/C}\left(c, c/C, a/A\right)\right]. \tag{12}$$

This equation shows that the *total* marginal utility of c, $V_c = u_c + C^{-1}u_{c/C}$, that takes into account that a change in c affects both absolute and relative consumption, can be represented as the product of the marginal utility of absolute consumption u_c and the factor $1 + m^{c/C}$.

Moreover, both $m^{c/C}$ and $m^{a/A}$ appear in the transversality condition. The assumptions that $u_c > 0$ and $u_{c/C} \geq 0$ together with the first-order condition (5) imply that $\lambda\left(t\right) > 0$ for $t \geq 0$. Hence, integration of (10) shows that the transversality condition (7) is equivalent to

$$\lim_{t\to\infty} \exp\left\{-\int_0^t \left[r\left(v\right) + \frac{m^{a/A}\left(v\right)}{1 + m^{c/C}\left(v\right)} \times \frac{c\left(v\right)}{a\left(v\right)}\right] dv\right\} a\left(t\right) = 0, \tag{13}$$

where $m^x\left(v\right) = m^x\left(c\left(v\right), c\left(v\right)/C\left(v\right), a\left(v\right)/A\left(v\right)\right)$ for $x = c/C$ and a/A. This modified transversality condition differs from its counterpart in the standard model in that the market rate of return is replaced by the effective rate of return. However, if relative wealth is irrelevant for utility so that $m^{a/A} = 0$, then the effective rate of return simplifies to the market rate of return. Consequently, if $m^{a/A} = 0$, then the transversality condition equals its counterpart of the standard model even if relative consumption matters for utility so that $m^{c/C} > 0$.

2.2 Production

There is a continuum of firms with mass 1. To allow for the occurrence of long-run endogenous growth in the simplest way possible, we follow Barro and Sala-i-Martin (1995, Subsection 4.3), which is inspired by Arrow (1962) and Romer (1986). The production function of the representative firm is given by $y = f\left(k, Bl\right)$, where y is output, k refers to input of physical capital, l denotes labor input, B is an index of knowledge available to the firm, and Bl denotes effective labor input. The assumptions made by Barro and Sala-i-Martin (1995) allow them to set $B = K$, where in this context K is interpreted as the aggregate capital stock of the economy. This simple production structure leads to a model that is – with respect to all aspects that are relevant for our analysis – isomorphic to more sophisticated models in which long-run economic growth is endogenously explained by purposeful R&D investments (see, for example, Romer, 1990).[3]

The production function has the standard neoclassical properties of i) positive and diminishing marginal products with respect to each input, ii) constant returns to scale, and iii) they fulfill the Inada conditions. There is perfect competition in all markets and the representative

[3]The effects of relative wealth preferences in the Romer (1990) model are analyzed in Hof and Prettner (2019). In the working paper version of this article (Hof and Prettner, 2016), relative consumption preferences are analyzed, too.

firm maximizes its profit by optimally choosing capital input k and labor input l, with the services of these two production factors being rented from households. Since there is a continuum of firms, the representative firm takes not only the rental rate of capital r and the real wage w, but also the available stock of knowledge $B\,(=K)$ as given. The corresponding first order conditions for a profit maximum can be written as

$$r = f_k\left(k, Kl\right), \qquad w = f_{(Bl)}\left(k, Kl\right)K, \tag{14}$$

where f_k and $f_{(Bl)}$ denote the marginal products of k and effective labor input Bl, respectively. Hence, $f_{(Bl)} \times B$ gives the marginal product of l. The conditions given by (14) require that each input is utilized up to the point at which its marginal product equals its real price.

3 The decentralized solution – General results of the fundamental factor approach

3.1 General features of the symmetric macroeconomic equilibrium

We follow the status literature with homogeneous individuals in which it is common practice to proceed with the analysis of the symmetric macroeconomic equilibrium.[4] Since both the mass of households and the mass of firms are normalized to unity, aggregate values of output, capital, labor, consumption, and wealth equal the corresponding average values denoted by Y, K, L, C, and A. In other words, aggregate and average values can be used interchangeably. In a macroeconomic equilibrium in which all markets clear, the rental rate r and the real wage w are endogenously determined by the following equations (for a detailed proof see Appendix B.1):

$$r = f_k\left(1, L\right), \qquad w = f_{(Bl)}\left(1, L\right)K. \tag{15}$$

Please note that L is treated as given, since we restrict our attention to the case in which labor supply is exogenous.

By assumption, private households are identical in every respect. Hence, in any symmetric macroeconomic equilibrium they make identical choices. Net loans of any household to other households and to firms are zero so that physical capital is the only store of households' wealth. Consequently, we have

$$c = C, \qquad a = A = K, \qquad l = L. \tag{16}$$

Substituting (16) into the flow budget constraint of the representative household (1) and taking into account that $rK + wL = Y = f\left(1, L\right)K$ holds due to constant returns to scale, we can show that the economy-wide resource constraint is given by $\dot{K} = Y - C = f\left(1, L\right)K - C$. Thus, the dynamic evolution of aggregate capital K is governed by the following differential equation (for a detailed proof see Appendix B.2):

$$\dot{K}/K = f\left(1, L\right) - C/K. \tag{17}$$

[4] Carroll et al. (1997) and Ljungqvist and Uhlig (2000) are exceptions.

Substituting (15) and (16) into (12) and (10), we obtain

$$\lambda = u_c(C,1,1)\left[1+m^{c/C}(C,1,1)\right], \tag{18}$$

$$\frac{\dot{\lambda}}{\lambda} = -\left[f_k(1,L)+\frac{m^{a/A}(C,1,1)}{1+m^{c/C}(C,1,1)}\times\frac{C}{K}-\rho\right]. \tag{19}$$

We carry the analysis out in the control-state space in which attention is focused on the behavior of aggregate consumption C and aggregate capital K by substituting for the shadow price of wealth λ. In this approach, the Euler equation for aggregate consumption in the symmetric macroeconomic equilibrium plays a decisive role. Differentiating (18) with respect to time t, substituting the resulting expression for $\dot{\lambda}/\lambda$ into (19), and solving for $\dot{C}/C$ we obtain

$$\frac{\dot{C}}{C} = \sigma^D(C)\left[f_k(1,L)+\eta^D(C)\times\frac{C}{K}-\rho\right], \tag{20}$$

where

$$\sigma^D(C) \equiv -\left[\varepsilon^{u_c,c}(C,1,1)+\frac{m^{c/C}(C,1,1)}{1+m^{c/C}(C,1,1)}\times\varepsilon^{m^{c/C},c}(C,1,1)\right]^{-1} \tag{21}$$

denotes the *effective elasticity of intertemporal substitution* in the decentralized economy (the superscript "D" stands for "Decentralized"), while

$$\eta^D(C) \equiv \frac{m^{a/A}(C,1,1)}{1+m^{c/C}(C,1,1)} \tag{22}$$

denotes the *comparison-induced extra return factor*. In (21), $\varepsilon^{u_c,c} \equiv cu_{cc}/u_c$ and $\varepsilon^{m^{c/C},c} \equiv cm_c^{c/C}/m^{c/C}$ are the elasticities of the marginal utility of absolute consumption u_c and of the percentage-MRS of relative consumption for absolute consumption $m^{c/C}$ with respect to absolute consumption c. Here, both elasticities are evaluated at $(c,c/C,a/A)=(C,1,1)$ due to the consideration of a symmetric macroeconomic equilibrium. The factor $\eta^D(C)$ captures the direct effects of both relative consumption and relative wealth preferences on the comparison-induced extra return given by $\eta^D(C)\times(C/K)$. It will become obvious that these preferences also exert an indirect effect by influencing the equilibrium level of the consumption-capital ratio C/K. Please note that neither the relative consumption motive nor the relative wealth motive affect the market rate of return $f_k(1,L)$, since we restrict our attention to the case in which labor supply is exogenously given.

Finally, substituting (15) and (16) into (13) and using (22) it is easily verified that in a symmetric macroeconomic equilibrium the transversality condition can be written as

$$\lim_{t\to\infty}\exp\left\{-\int_0^t\left[f_k(1,L)+\eta^D(C(v))\times\frac{C(v)}{K(v)}\right]dv\right\}K(t)=0. \tag{23}$$

Hence, in the control-state space analysis, the dynamic evolution of C and K is governed by the differential equations (17) and (20), the transversality condition (23), and the initial condition $K(0)=K_0$.

The differential equations (17) and (20) contain the terms $f(1,L)$ and $f_k(1,L)$, respectively. For various results derived in the rest of the paper it is of crucial importance that (for a proof see Appendix B.3)

$$f(1,L) > f_k(1,L). \tag{24}$$

For given employment L, $f_k(1,L)$ gives the *constant* value of the private marginal product of capital in the decentralized equilibrium. In the decentralized economy, the expression $f(1,L) = Y/K$ has a single meaning: it describes the *constant* average product of capital (i.e., the ratio of aggregate production Y to aggregate capital K). In the socially planned economy discussed in section 5, $f(1,L)$ also represents the social marginal product of capital, i.e., the marginal product as perceived by the social planner that internalizes the knowledge spillovers resulting from the capital accumulation of individual firms.

3.2 Balanced growth path (BGP) – Existence and Properties

In the following we provide sufficient conditions for the existence of an economically meaningful BGP in the decentralized economy and analyze its properties. Henceforth, we use the term "economically meaningful BGP" to describe a BGP in which 1) the growth rate is strictly positive, 2) the consumption-capital ratio is strictly positive, and 3) the transversality condition is satisfied.

Proposition 1. *(Decentralized BGP – The roles of σ^D and η^D)*

A) If i) the specification of the instantaneous utility function $u = u(c, c/C, a/A)$ has the property that both the decentralized effective elasticity of intertemporal substitution and the comparison-induced extra return factor are independent of C such that

$$\sigma^D(C) = \hat{\sigma}, \qquad \eta^D(C) = \hat{\eta}, \qquad \forall C > 0, \tag{25}$$

where $\hat{\sigma} > 0$ and $\hat{\eta} \geq 0$ are constants and ii) the condition

$$\max\left\{\frac{[1-(1/\hat{\sigma})]\,[f_k(1,L) + \hat{\eta} f(1,L)]}{1+\hat{\eta}}, 0\right\} < \rho < f_k(1,L) + \hat{\eta} f(1,L) \tag{26}$$

is satisfied, then an economically meaningful BGP exists in the decentralized economy. Along the BGP the common growth rate of consumption and capital $g^D = (\dot{C}/C)^D = (\dot{K}/K)^D$ and the consumption-capital ratio $(C/K)^D$ are given by:

$$g^D = \frac{f_k(1,L) - \rho + \hat{\eta} f(1,L)}{(1/\hat{\sigma}) + \hat{\eta}} > 0, \tag{27}$$

$$(C/K)^D = \frac{(1/\hat{\sigma}) f(1,L) - [f_k(1,L) - \rho]}{(1/\hat{\sigma}) + \hat{\eta}} > 0. \tag{28}$$

B) The model has no transitional dynamics.

For a proof of Proposition 1 see Appendix B.4. From Equations (27)–(28) it is obvious that the common growth rate g^D and the consumption-capital ratio $(C/K)^D$ are completely determined by the following five mathematical expressions/parameters: $f(1, L)$, $f_k(1, L)$, ρ, $\hat{\sigma}$, and $\hat{\eta}$. Hence, the same is true for the comparison-induced extra return $\hat{\eta} \times (C/K)^D$. Since labor supply L is exogenously given by assumption, relative consumption and relative wealth preferences affect the BGP, if at all, only via the $\hat{\sigma}$-channel and/or the $\hat{\eta}$-channel. For this reason, it is very important to thoroughly understand the operation of these two channels. The following proposition and the subsequent interpretation provide the relevant details.

Proposition 2. *(The dependence of the decentralized BGP on $\hat{\sigma}$ and $\hat{\eta}$)*

In the decentralized economy the growth rate g^D depends positively on both the effective elasticity of intertemporal substitution $\hat{\sigma}$ and the comparison-induced extra return factor $\hat{\eta}$, while the opposite results obtain for the consumption-capital ratio $(C/K)^D$.

The comparison-induced extra return $\hat{\eta} \times (C/K)^D$ depends positively on $\hat{\eta}$. Moreover, if $\hat{\eta} > 0$, then it depends negatively on $\hat{\sigma}$. However, if $\hat{\eta} = 0$, then it is independent of $\hat{\sigma}$ because $\hat{\eta} \times (C/K)^D = 0$ holds for all $\hat{\sigma} > 0$. In mathematical terms, we have

$$\frac{\partial g^D}{\partial \hat{\sigma}} > 0, \quad \frac{\partial (C/K)^D}{\partial \hat{\sigma}} < 0, \quad sgn\left(\frac{\partial \left[\hat{\eta} \times (C/K)^D\right]}{\partial \hat{\sigma}}\right) = -sgn\left(\hat{\eta}\right), \tag{29}$$

$$\frac{\partial g^D}{\partial \hat{\eta}} > 0, \quad \frac{\partial (C/K)^D}{\partial \hat{\eta}} < 0, \quad \frac{\partial \left[\hat{\eta} \times (C/K)^D\right]}{\partial \hat{\eta}} > 0. \tag{30}$$

For a proof of the mathematical assertions made in (29) and (30) see Appendix B.5. The economic interpretation of the implied qualitative dependence of the BGP on $\hat{\sigma}$ and $\hat{\eta}$ is mainly based on the equation

$$g^D = \hat{\sigma}\left[f_k(1, L) + \hat{\eta} \times (C/K)^D - \rho\right] \tag{31}$$

that is obtained by substituting $(\dot{C}/C)^D = g^D$ into the steady-state version of the Euler equation for aggregate consumption given by (20).

First, we provide the interpretation for the dependence of the BGP on $\hat{\sigma}$. According to (31) a ceteris paribus increase in the effective elasticity of intertemporal substitution $\hat{\sigma}$ exerts a direct effect on the decentralized growth rate g^D and, if $\hat{\eta} > 0$ holds, also an indirect effect via the reaction of $(C/K)^D$. The direct effect results from the fact that a rise in $\hat{\sigma}$ increases the willingness of private households to substitute future absolute consumption for present absolute consumption. In other words, there is an increase in the willingness to save which, in turn, causes the common rate of growth of aggregate capital, consumption, and output to rise. If $\hat{\eta} > 0$ holds, there is also an indirect effect. The decrease in the aggregate consumption-capital ratio $(C/K)^D$ that results from the rise in $\hat{\sigma}$ causes the comparison-induced extra return $\hat{\eta} \times (C/K)^D$ and, hence, the effective rate of return, $f_k(1, L) + \hat{\eta} \times (C/K)^D$, to fall. The latter effect dampens the incentives to save and thus exerts a negative effect on the accumulation of capital. Since the positive direct effect exceeds the negative indirect effect, the decentralized growth rate g^D depends positively on $\hat{\sigma}$.

Second, we explain the dependence of the BGP on $\hat{\eta}$. A ceteris paribus rise in the comparison-induced extra return factor $\hat{\eta}$ causes the extra return $\hat{\eta} \times (C/K)^D$ to increase, because the rise in $\hat{\eta}$ is only partially offset by the fall in $(C/K)^D$. The resulting rise in the effective rate of return, $f_k(1, L) + \hat{\eta} \times (C/K)^D$, enhances the incentives to save and thus boosts the accumulation of capital. Hence, the decentralized growth rate g^D depends positively on $\hat{\eta}$.

In the next two propositions that build on the results given above we dig deeper by considering explicitly both the strength of the relative consumption motive and the strength of the relative wealth motive. In this context the definitions of $\sigma^D(C)$ and $\eta^D(C)$ given by (21) and (22) play a crucial role. Henceforth, we use the term "symmetric situations" to describe situations in which $c = C$ and $a = A$ hold so that $c/C = 1$ and $a/A = 1$.

Proposition 3. *(Decentralized BGP – The roles of $m^{c/C}$, $m^{a/A}$, and $|\varepsilon^{u_c,c}|$)*

If the instantaneous utility function $u = u(c, c/C, a/A)$ exhibits the property that in symmetric situations $m^{c/C}$, $m^{a/A}$, and $\varepsilon^{u_c,c}$ are constant functions of C so that

$$m^{c/C}(C,1,1) = \hat{m}^{c/C}, \quad m^{a/A}(C,1,1) = \hat{m}^{a/A}, \quad \varepsilon^{u_c,c}(C,1,1) = \hat{\varepsilon}^{u_c,c}, \quad \forall C > 0, \tag{32}$$

where $\hat{m}^{c/C} \geq 0$, $\hat{m}^{a/A} \geq 0$ (with $\max\{\hat{m}^{c/C}, \hat{m}^{a/A}\} > 0$), and $\hat{\varepsilon}^{u_c,c} < 0$ are constants, then the two conditions given in (25) [Proposition 1] are satisfied, since

$$\sigma^D(C) = \frac{1}{|\hat{\varepsilon}^{u_c,c}|} \equiv \hat{\sigma}, \qquad \eta^D(C) = \frac{\hat{m}^{a/A}}{1 + \hat{m}^{c/C}} \equiv \hat{\eta}, \qquad \forall C > 0. \tag{33}$$

If, in addition, the condition (26) given in Proposition 1 is satisfied for the values of $\hat{\sigma}$ and $\hat{\eta}$ defined by (33), then an economically meaningful decentralized BGP exists. The corresponding BGP growth rate is given by

$$g^D = \frac{f_k(1,L) - \rho + \dfrac{\hat{m}^{a/A}}{1 + \hat{m}^{c/C}} \times f(1,L)}{|\hat{\varepsilon}^{u_c,c}| + \dfrac{\hat{m}^{a/A}}{1 + \hat{m}^{c/C}}}. \tag{34}$$

For a proof of Proposition 3 see Appendix B.6. In general, the constant $\hat{\eta}$ depends on both $\hat{m}^{c/C}$ and $\hat{m}^{a/A}$, where the latter two constants measure the strength of the relative consumption motive and the relative wealth motive, respectively, in symmetric situations. Consequently, $\hat{m}^{c/C}$ and $\hat{m}^{a/A}$ yield only *local* information about the strength of the corresponding motives. Below we discuss six specifications of the instantaneous utility function $u = u(c, c/C, a/A)$ in which the condition (32) is satisfied. In four out of these six illustrations the functions $m^{c/C}(c, c/C, a/A)$ and $m^{a/A}(c, c/C, a/A)$ are constant functions over their whole domains so that $\hat{m}^{c/C}$ and $\hat{m}^{a/A}$ are also measures of the *global* strength of the relative consumption motive and the relative wealth motive, respectively.

The constant $\hat{\sigma} = 1/|\hat{\varepsilon}^{u_c,c}|$ measures the willingness to substitute absolute consumption intertemporally in *symmetric situations*. It depends neither on $\hat{m}^{c/C}$ nor on $\hat{m}^{a/A}$. While according to (21) the independence of σ^D on the strength of the relative wealth motive is a general property of the model, the irrelevance of $\hat{m}^{c/C}$ for $\hat{\sigma}$ results from an assumption that we made in (32) to ensure the existence of a BGP, namely that $m^{c/C}(C,1,1)$ is a constant function of C. For details see the proof of Proposition 3.

Equation (34) plays a decisive role in the rest of the paper. It shows that the decentralized growth rate g^D can be *ultimately* represented as a function of $f(1,L)$, $f_k(1,L)$, ρ, $\hat{m}^{a/A}$, $\hat{m}^{c/C}$, and $|\hat{\varepsilon}^{u_c,c}|$. Henceforth, these six terms are called the "fundamental factors" of growth in the decentralized economy. Obviously, these fundamental factors are also the *ultimate* determinants of the associated consumption-capital ratio $(C/K)^D = f(1, L) - g^D$ and the resulting comparison-induced extra return $\hat{\eta} \times (C/K)^D$. In the following proposition we analyze the implications of *ceteris paribus* changes in the three fundamental factors that depend on the specification of the instantaneous utility function $u = u(c, c/C, a/A)$, namely $\hat{m}^{a/A}$, $\hat{m}^{c/C}$, and $|\hat{\varepsilon}^{u_c,c}|$.

Proposition 4. *(The dependence of the decentralized growth rate g^D given by Equation (34) on the fundamental factors $\hat{m}^{a/A}$, $\hat{m}^{c/C}$, and $|\hat{\varepsilon}^{u_c,c}|$)*

i) g^D depends positively on the strength of the relative wealth motive (in symmetric situations) as measured by $\hat{m}^{a/A}$, where $\hat{m}^{a/A}$ affects g^D exclusively via the $\hat{\eta}$-channel:

$$\frac{\partial g^D}{\partial \hat{m}^{a/A}} = \frac{\partial g^D}{\partial \hat{\eta}} \times \frac{\partial \hat{\eta}}{\partial \hat{m}^{a/A}} = \frac{\partial g^D}{\partial \hat{\eta}} \times \frac{1}{1 + \hat{m}^{c/C}} > 0. \tag{35}$$

ii) If relative wealth matters for utility so that $\hat{m}^{a/A} > 0$, then g^D depends negatively on the strength of the relative consumption motive (in symmetric situations) as measured by $\hat{m}^{c/C}$, where $\hat{m}^{c/C}$ affects g^D exclusively via the $\hat{\eta}$-channel. However, if relative wealth is irrelevant for utility so that $\hat{m}^{a/A} = 0$ and, hence, $\hat{\eta} = 0$, then g^D is independent of $\hat{m}^{c/C}$:

$$\begin{aligned} \frac{\partial g^D}{\partial \hat{m}^{c/C}} &= \frac{\partial g^D}{\partial \hat{\eta}} \times \frac{\partial \hat{\eta}}{\partial \hat{m}^{c/C}} = -\frac{\partial g^D}{\partial \hat{\eta}} \times \frac{\hat{m}^{a/A}}{\left(1 + \hat{m}^{c/C}\right)^2} \leq 0 \\ &\Rightarrow \quad sgn\left(\frac{\partial g^D}{\partial \hat{m}^{c/C}}\right) = -sgn\left(\hat{m}^{a/A}\right). \end{aligned} \tag{36}$$

iii) g^D depends negatively on the absolute value of the elasticity of the marginal utility of absolute consumption with respect to absolute consumption (in symmetric situations) $|\hat{\varepsilon}^{u_c,c}|$, where $|\hat{\varepsilon}^{u_c,c}|$ affects g^D exclusively via the $\hat{\sigma}$-channel:

$$\frac{\partial g^D}{\partial |\hat{\varepsilon}^{u_c,c}|} = \frac{\partial g^D}{\partial \hat{\sigma}} \times \frac{\partial \hat{\sigma}}{\partial |\hat{\varepsilon}^{u_c,c}|} = -\frac{\partial g^D}{\partial \hat{\sigma}} \times \frac{1}{|\hat{\varepsilon}^{u_c,c}|^2} < 0. \tag{37}$$

The mathematical results given in (35)–(37) are easily obtained by 1) using the chain rule of differentiation, 2) calculating the partial derivatives of $\hat{\eta} = \hat{m}^{a/A} / \left(1 + \hat{m}^{c/C}\right)$ with respect to $\hat{m}^{a/A}$ and $\hat{m}^{c/C}$, and the derivative of $\hat{\sigma} = 1/|\hat{\varepsilon}^{u_c,c}|$ with respect to $|\hat{\varepsilon}^{u_c,c}|$, and 3) making use of the fact that $\partial g^D/\partial \hat{\sigma} > 0$ and $\partial g^D/\partial \hat{\eta} > 0$ hold according to Proposition 2. The economic interpretation of the results given in Proposition 4 is straightforward.

i) A ceteris paribus increase in the constant $\hat{m}^{a/A}$ that measures the strength of the relative wealth motive (in symmetric situations) influences the common growth rate g^D exclusively via the resulting increase in the comparison-induced extra return parameter $\hat{\eta}$. According to Proposition 2 the rise in $\hat{\eta}$ causes the comparison-induced extra return $\hat{\eta} \times (C/K)^D$ to increase, because the rise in $\hat{\eta}$ is only partially offset by the fall in $(C/K)^D$. The resulting rise in the effective rate of return, $f_k(1, L) + \hat{\eta} \times (C/K)^D$, enhances the incentives to save and thus boosts economic growth. Hence, g^D depends positively on $\hat{m}^{a/A}$.

ii) A ceteris paribus increase in the constant $\hat{m}^{c/C}$ that measures the strength of the relative consumption motive (in symmetric situations) affects the common growth rate g^D, if at all, solely via its influence on the comparison-induced extra return parameter $\hat{\eta}$. We have to distinguish between two cases. Case 1: If relative wealth matters for utility so that $\hat{m}^{a/A} > 0$, then a rise in $\hat{m}^{c/C}$ causes $\hat{\eta}$ to decrease. According to Proposition 2, the fall in $\hat{\eta}$ causes the comparison-induced extra return $\hat{\eta} \times (C/K)^D$ to decrease. The resulting fall in the effective rate of return, $f_k(1, L) + \hat{\eta} \times (C/K)^D$, attenuates the incentives to save and thus reduces economic growth. Hence, g^D depends negatively on $\hat{m}^{c/C}$. Case 2: If relative wealth is irrelevant for utility so that $\hat{m}^{a/A} = 0$ and $\hat{\eta} = 0$, then changes in $\hat{m}^{c/C}$ do not affect the decentralized growth rate g^D, because in this case not only $\hat{\sigma}$ but also $\hat{\eta}$ is independent of $\hat{m}^{c/C}$.

iii) A ceteris paribus change in the absolute value of the elasticity of the marginal utility of absolute consumption with respect to absolute consumption (in symmetric situations) $|\hat{\varepsilon}^{u_c,c}|$ affects the decentralized growth rate exclusively via the resulting change in the effective elasticity of intertemporal substitution $\hat{\sigma}$: A rise in $|\hat{\varepsilon}^{u_c,c}|$ causes $\hat{\sigma}$ to decrease which, according to Proposition 2, leads to a fall in g^D.

Propositions 3 and 4 are crucial results that are used several times below to derive further insights.[5] Proposition 3 yields a representation of the decentralized growth rate that is robust with respect to the specification of the instantaneous utility function as long as $u = u(c, c/C, a/A)$ satisfies the quite weak conditions that are given in (32) [Proposition 3]. Proposition 4 allows to study the effects of *ceteris paribus* changes in the strength of the relative consumption and relative wealth motives. We show below that there are instances in which such ceteris paribus thought experiments cannot be carried out within the standard approach that does not identify the fundamental factors but restricts attention to the parameters of the utility function. Consequently, the standard analysis is prone to erroneous conclusions whenever it employs utility functions in which the change in a single parameter does not constitute a change in a single fundamental factor. For example, a prominent specification used in the status literature has the property that the parameter that *seems* to determine the strength of the relative consumption motive *actually* affects not only $\hat{m}^{c/C}$, but also $\hat{m}^{a/A}$ and $|\hat{\varepsilon}^{u_c,c}|$.

After having presented these four propositions, the crucial question is whether there exist specifications of $u = u(c, c/C, a/A)$ that satisfy the sufficient conditions for the existence of a BGP given by (25) in Proposition 1 and (32) in Proposition 3. The following proposition answers this question in the affirmative. It presents a quite general specification of u that encompasses several prominent specifications used in the literature.

Proposition 5. *(A general specification of $u(c, c/C, a/A)$ that ensures the existence of an economically meaningful BGP)*

Let the instantaneous utility function have the form

$$u(c, c/C, a/A) = \frac{1}{1-\theta}\left\{\left[c^{\xi_1} Q(c/C, a/A)\right]^{1-\theta} - 1\right\}, \tag{38}$$

where ξ_1 and θ are parameters, while $Q(\cdot, \cdot)$ denotes a function.

[5] The results given in i) and ii) of Proposition 4 are consistent with those derived in the seminal contribution by Tournemaine and Tsoukis (2008) for the case of an *exogenously* given labor supply. We go beyond their framework by 1) introducing the concept of the fundamental factors, 2) identifying utility functions (that play a prominent role in the status literature) in which the standard approach is prone to erroneous conclusions, 3) considering also the socially planned economy, and 4) offering alternative economic interpretations.

A) If the parameters θ and ξ_1 satisfy the conditions

$$\xi_1 > 0, \quad \theta > 0, \quad 1 + (\theta - 1)\,\xi_1 > 0 \tag{39}$$

and, in addition, the function $Q\,(c/C, a/A)$ exhibits the property that

$$Q > 0, \quad Q_{c/C} \geq 0, \quad Q_{a/A} \geq 0, \quad Q_{c/C} > 0 \vee Q_{a/A} > 0 \tag{40}$$

hold over its domain denoted by Θ_Q, then the instantaneous utility function given by (38) is well-behaved in the sense that all assumptions made in (2) are satisfied.

B) Let $\varepsilon^{Q,c/C} \equiv (c/C)\,Q_{c/C}/Q$, and $\varepsilon^{Q,a/A} \equiv (a/A)\,Q_{a/A}/Q$ denote the elasticities of the function $Q\,(c/C, a/A)$ with respect to c/C and a/A, respectively. The instantaneous utility function given by (38) satisfies the three conditions given in (32) [Proposition 3], since

$$m^{c/C}\,(C,1,1) = \hat{m}^{c/C}, \quad m^{a/A}\,(C,1,1) = \hat{m}^{a/A}, \quad \varepsilon^{u_c,c}\,(C,1,1) = \hat{\varepsilon}^{u_c,c}$$

hold for $C > 0$, where

$$\hat{m}^{c/C} = \frac{\hat{\varepsilon}^{Q,c/C}}{\xi_1} \geq 0, \quad \hat{m}^{a/A} = \frac{\hat{\varepsilon}^{Q,a/A}}{\xi_1} \geq 0, \quad \hat{\varepsilon}^{u_c,c} = -\left[1 + (\theta - 1)\,\xi_1\right] < 0, \tag{41}$$

with $\hat{\varepsilon}^{Q,c/C} \equiv \varepsilon^{Q,c/C}\,(1,1)$ and $\hat{\varepsilon}^{Q,a/A} \equiv \varepsilon^{Q,a/A}\,(1,1)$. Consequently, the two conditions given in (25) [Proposition 1] are satisfied, too:

$$\sigma^D\,(C) = \frac{1}{1 + (\theta - 1)\,\xi_1} \equiv \hat{\sigma} > 0, \quad \eta^D\,(C) = \frac{\hat{\varepsilon}^{Q,a/A}/\xi_1}{1 + \hat{\varepsilon}^{Q,c/C}/\xi_1} \equiv \hat{\eta} \geq 0, \quad \forall C > 0. \tag{42}$$

If the constants $\hat{\sigma}$ and $\hat{\eta}$ defined by (42) satisfy the condition (26) given in Proposition 1, then an economically meaningful decentralized BGP exists. The corresponding constant common growth rate is given by

$$g^D = \frac{f_k\,(1,L) - \rho + \dfrac{\hat{\varepsilon}^{Q,a/A}/\xi_1}{1 + \hat{\varepsilon}^{Q,c/C}/\xi_1} \times f\,(1,L)}{1 + (\theta - 1)\,\xi_1 + \dfrac{\hat{\varepsilon}^{Q,a/A}/\xi_1}{1 + \hat{\varepsilon}^{Q,c/C}/\xi_1}} > 0. \tag{43}$$

In Appendix B.7, we provide a proof for an extended version of Proposition 5. The extended version also contains the complicated conditions that the specification of u given by (38) has to satisfy so that the alternative representation of preferences defined by (3), $V\,(c, C, a, A) \equiv u\,(c, c/C, a/A)$, is well-behaved in the sense that all assumptions made in (4) are satisfied. Moreover, it makes explicit the weak restrictions that we impose on $u = u\,(c, c/C, a/A)$ in Proposition 5 from the outset as well as the additional restrictions that need to be imposed to ensure the existence of a BGP.[6]

[6] For the sake of robustness of the results, we intend to introduce initial restrictions with respect to the specification of the instantaneous utility function u that are as weak as possible. The specification given by (38) is obtained by considering the general case in which u results from the transformation of a multiplicatively

In the next section we show that several specifications used in the literature can be interpreted as special cases of (38). Sometimes it will be even adequate to use the simplified version of (38) that is obtained by assuming that $Q(c/C, a/A) = (c/C)^{\xi_2} (a/A)^{\xi_3}$. Please note that in this special case the elasticities of the function Q are constant functions, i.e., $\varepsilon^{Q,c/C}(c/C, a/A) = \xi_2$ and $\varepsilon^{Q,a/A}(c/C, a/A) = \xi_3$ hold for all $(c/C, a/A) \in \Theta_Q$. Further implications of this specification of $Q(c/C, a/A)$ are summarized in the following corollary.

Corollary 1. *(A simplified version of the general specification of the utility function (38) – properties and implications for the BGP growth rate)*

Let the instantaneous utility function take the form

$$u(c, c/C, a/A) = \frac{1}{1-\theta}\left\{\left[c^{\xi_1}(c/C)^{\xi_2}(a/A)^{\xi_3}\right]^{1-\theta} - 1\right\}, \tag{44}$$

where

$$\theta > 0, \quad \xi_1 > 0, \quad \xi_2 \geq 0, \quad \xi_3 \geq 0, \quad \max\{\xi_2, \xi_3\} > 0, \tag{45}$$

$$(1-\theta)(\xi_1 + \xi_2 + \xi_3) < 1. \tag{46}$$

A) *The instantaneous utility function (44) and the resulting representation of preferences given by $V(c, C, a, A) \equiv u(c, c/C, a/A)$ are well-behaved in the sense that all assumptions made in (2) and (4) are satisfied.*

B) *The expressions for $\hat{m}^{c/C}$, $\hat{m}^{a/A}$, and $\hat{\eta}$ given in (41) [Proposition 5] simplify to*

$$\hat{m}^{c/C} = \xi_2/\xi_1 \geq 0, \quad \hat{m}^{a/A} = \xi_3/\xi_1 \geq 0, \quad \hat{\eta} = \frac{\xi_3/\xi_1}{1 + \xi_2/\xi_1} \geq 0, \tag{47}$$

where $\max\{\hat{m}^{c/C}, \hat{m}^{a/A}\} > 0$, *while the expressions for $\hat{\varepsilon}^{u_c,c}$ and $\hat{\sigma}$ given in (42) [Proposition 5] remain unchanged:*

$$\hat{\varepsilon}^{u_c,c} = -\left[1 + (\theta - 1)\xi_1\right] < 0, \quad \hat{\sigma} = \frac{1}{1 + (\theta - 1)\xi_1} > 0. \tag{48}$$

C) *The expression for the common growth rate given in (43) [Proposition 5] simplifies to*

$$g^D = \frac{f_k(1, L) - \rho + \dfrac{\xi_3/\xi_1}{1 + \xi_2/\xi_1} f(1, L)}{1 + (\theta - 1)\xi_1 + \dfrac{\xi_3/\xi_1}{1 + \xi_2/\xi_1}}. \tag{49}$$

The equations (47), (48), and (49) given in B) and C) are easily obtained by substituting $\hat{\varepsilon}^{Q,c/C} = \xi_2$ and $\hat{\varepsilon}^{Q,a/A} = \xi_3$ into (41), (42), and (43) given in Proposition 5. By contrast, the proof of A) requires tedious calculations that are provided in Appendix B.8. Please note that under the specification (44) the constants $\hat{m}^{c/C}$ $(= \xi_2/\xi_1)$ and $\hat{m}^{a/A}$ $(= \xi_3/\xi_1)$ measure the strength of the relative consumption motive and the relative wealth motive not only locally in

separable function in the sense that $u(c, c/C, a/A) = T[P(c)Q(c/C, a/A)]$, and showing that a BGP exists if and only if the transformation T is of the CRRA type and $P(c)$ is a power function.

symmetric situations, but *globally.* This follows from the fact that both $m^{c/C}(c, c/C, a/A)$ and $m^{a/A}(c, c/C, a/A)$ are constant functions over their whole domain.

In the next section we reexamine various specifications of relative consumption and relative wealth preferences that are employed in the literature. All these specifications can be expressed as special cases of (38) and (44), respectively. In this context, we put special emphasis on providing explanations for the potential fallacies of the existing literature that result from its ignorance of the fundamental factors. In this context, the exponent of absolute consumption given by the parameter ξ_1 plays a decisive role. We show that using the simplifying assumption that $\xi_1 = 1$ is not at all innocuous, because it implies a significant loss of generality. More precisely, if $\xi_1 = 1$, then the general specification (38) does not encompass one of the most prominent specifications in the literature.

4 The decentralized solution – Utility functions used in the literature and potential fallacies of the standard analysis

4.1 Preliminaries

One of the main goals of this section is to show that the traditional method of analysis involves the great risk of drawing erroneous conclusions. For instance, it will become obvious that one of the most prominent specifications of the instantaneous utility function used in the literature seems to allow for the possibility that relative consumption preferences enhance long-run growth. Using our fundamental factor approach we show that such a conclusion is clearly at variance with our Proposition 4 given above and is seriously flawed. For the explanation of this fallacy the following properties of (38) [resp. (44)] that follow directly from Proposition 5 and Corollary 1 are decisive:

Corollary 2. *(The effects of changes in the parameters of the utility functions (38) and (44) on the fundamental factors)*

A) Ceteris paribus changes in $\hat{\varepsilon}^{Q,c/C} \equiv \varepsilon^{Q,c/C}(1,1)$ *[resp.* ξ_2*] affect only the percentage-MRS of relative consumption* $\hat{m}^{c/C}$*.*

B) Ceteris paribus changes in $\hat{\varepsilon}^{Q,a/A} \equiv \varepsilon^{Q,a/A}(1,1)$ *[resp.* ξ_3*] exert solely an effect on the percentage-MRS of relative wealth* $\hat{m}^{a/A}$*.*

C) Ceteris paribus changes in θ *influence exclusively the absolute value of the elasticity of marginal utility of absolute consumption with respect to absolute consumption,* $|\hat{\varepsilon}^{u_c,c}|$*.*

D) In contrast to $\hat{\varepsilon}^{Q,c/C}$*,* $\hat{\varepsilon}^{Q,a/A}$*, and* θ *[resp.* ξ_2*,* ξ_3*, and* θ*], ceteris paribus changes in the exponent of absolute consumption,* ξ_1*, affect*

i) $\hat{m}^{c/C}$*, provided that* $\hat{\varepsilon}^{Q,c/C} \neq 0$ *[resp.* $\xi_2 \neq 0$*],*

ii) $\hat{m}^{a/A}$*, provided that* $\hat{\varepsilon}^{Q,a/A} \neq 0$ *[resp.* $\xi_3 \neq 0$*] and,*

iii) $|\hat{\varepsilon}^{u_c,c}|$*, provided that* $\theta \neq 1$*.*

E) In the special case in which $\xi_1 = 1$ holds, we have

$$\hat{m}^{c/C} = \hat{\varepsilon}^{Q,c/C}, \qquad \hat{m}^{a/A} = \hat{\varepsilon}^{Q,a/A}, \qquad |\hat{\varepsilon}^{u_c,c}| = \theta, \qquad \textit{and} \tag{50}$$

$$\hat{m}^{c/C} = \xi_2, \qquad \hat{m}^{a/A} = \xi_3, \qquad |\hat{\varepsilon}^{u_c,c}| = \theta, \tag{51}$$

which, in turn, implies that there is a one-to-one correspondence between the three fundamental factors $\hat{m}^{c/C}$, $\hat{m}^{a/A}$, and $|\hat{\varepsilon}^{u_c,c}|$ and the three parameters given by $\hat{\varepsilon}^{Q,c/C}$, $\hat{\varepsilon}^{Q,a/A}$, and θ [resp. ξ_2, ξ_3, θ].

Point D) implies that ceteris paribus changes in ξ_1 may alter the willingness to substitute 1) relative consumption for absolute consumption, 2) relative wealth for absolute consumption, and 3) future absolute consumption for current absolute consumption. This property leads to complications and erroneous conclusions with respect to the implications of relative consumption and relative wealth preferences in case that a specification of the (44)-type is employed in which the parameters ξ_1, ξ_2, ξ_3, and θ are tied by a *functional dependence*.

A simple, but quite relevant example of functional dependence is given by the special case in which $\xi_1 = 1 - \xi_2 - \xi_3$ holds, where $\xi_2 > 0$ and $\xi_3 > 0$ are independent parameters. This specification implies that instantaneous utility depends on a geometric weighted average of absolute consumption, relative consumption, and relative wealth. The decisive feature of this specification is that any rise in the exponents of either relative consumption, ξ_2, or relative wealth, ξ_3, is inevitably associated with a fall in the exponent of absolute consumption, $\xi_1 = 1-\xi_2-\xi_3$. This property implies that a rise in the exponent of relative consumption, ξ_2, must not be interpreted as a ceteris paribus increase in the strength of the relative consumption motive that is measured correctly by the percentage-MRS $\hat{m}^{c/C} = \xi_2/\xi_1$. The fall in $\xi_1 = 1-\xi_2-\xi_3$ not only reinforces the increase in $\hat{m}^{c/C}$, but also raises the strength of the relative wealth motive as measured by the percentage-MRS $\hat{m}^{a/A} = \xi_3/\xi_1$ and, in addition, affects $|\hat{\varepsilon}^{u_c,c}|$ provided that $\theta \neq 1$. Hence, if $\xi_1 = 1 - \xi_2 - \xi_3$ holds, then a ceteris paribus rise in the exponent of relative consumption, ξ_2, is not equivalent to as a ceteris paribus increase in the strength of the relative consumption motive. The literature, however, seems to be unaware of this non-equivalence even in the pure relative consumption case in which $\xi_3 = 0$ and $\xi_1 = 1 - \xi_2$ hold.

We find it helpful to illustrate and elucidate the possibility of erroneous conclusions in the presence of a functional dependence of the parameters by means of a compact mathematical representation. To keep this representation simple without impairing the main message we consider the case in which i) the instantaneous utility function is of the form given by (44), and ii) both ξ_1 and ξ_3 depend on the exponent of relative consumption ξ_2, $\xi_1 = \xi_1(\xi_2)$, and $\xi_3 = \xi_3(\xi_2)$, while ξ_2 and θ are treated as independent parameters.[7] Taking into account that according to (34), g^D depends on the six fundamental factors $\hat{m}^{c/C}$, $\hat{m}^{a/A}$, $|\hat{\varepsilon}^{u_c,c}|$, $f(1,L)$, $f_k(1,L)$, and ρ, where the last three arguments are independent of ξ_2, and that according to (47) and (48), $\hat{m}^{c/C} = \xi_2/\xi_1$, $\hat{m}^{a/A} = \xi_3/\xi_1$, and $|\hat{\varepsilon}^{u_c,c}| = 1 + (\theta - 1)\xi_1$ hold, it is obvious that

[7]Things become more complicated if the specification of u exhibits the property that the parameter ξ_2 itself is a function of other parameters $\pi_1, \ldots, \pi_m$, $\xi_2 = \xi_2(\pi_1, \ldots, \pi_m)$, and the same is true for the parameters ξ_1, ξ_3, and θ. For details, see Appendix C.1.

the *total* derivative of the common growth rate with respect to ξ_2 is given by

$$\begin{aligned}
\frac{dg^D}{d\xi_2} &= \frac{\partial g^D}{\partial \hat{m}^{c/C}} \cdot \frac{d\hat{m}^{c/C}}{d\xi_2} + \frac{\partial g^D}{\partial \hat{m}^{a/A}} \cdot \frac{d\hat{m}^{a/A}}{d\xi_2} + \frac{\partial g^D}{\partial \left|\hat{\varepsilon}^{u_c,c}\right|} \cdot \frac{d\left|\hat{\varepsilon}^{u_c,c}\right|}{d\xi_2} \\
&= \frac{\partial g^D}{\partial \hat{m}^{c/C}} \left(\frac{1}{\xi_1} - \frac{\xi_2}{\xi_1^2} \frac{d\xi_1}{d\xi_2} \right) + \frac{\partial g^D}{\partial \hat{m}^{a/A}} \left(\frac{1}{\xi_1} \frac{d\xi_3}{d\xi_2} - \frac{\xi_3}{\xi_1^2} \frac{d\xi_1}{d\xi_2} \right) \\
&\quad + \frac{\partial g^D}{\partial \left|\hat{\varepsilon}^{u_c,c}\right|} (\theta - 1) \frac{d\xi_1}{d\xi_2}. \qquad (52)
\end{aligned}$$

Recall that according to our fundamental factor approach [see Proposition 4, Equation (36)], the correct measure to assess the effects of relative consumption preferences on the common growth rate g^D is given by the partial derivative $\partial g^D / \partial \hat{m}^{c/C}$. The standard approach, however, confines the analysis to the calculation of $dg^D/d\xi_2$. In this context, the standard approach is unaware of the decomposition given by the right-hand side of equation (52), which expresses in mathematical terms the idea described above that a change in the exponent of relative consumption ξ_2 might affect the common growth rate not only via the fundamental factor $\hat{m}^{c/C}$ but also via the fundamental factors $\hat{m}^{a/A}$ and $|\hat{\varepsilon}^{u_c,c}|$. If either ξ_1 or both ξ_1 and ξ_3 depend on ξ_2, then an analysis that uses the derivative $dg^D/d\xi_2$ to assess the effects of relative consumption preferences on the common growth rate involves two sources of error: First, the sign of $dg^D/d\xi_2$ may deviate from the sign of $\partial g^D/\partial \hat{m}^{c/C}$. While according to Proposition 4, $\partial g^D/\partial \hat{m}^{c/C} \leq 0$ holds for all instantaneous utility functions that satisfy (32), we show below that well-known specifications exist in which we cannot rule out that $dg^D/d\xi_2 > 0$ holds. This leads to the erroneous conclusion that relative consumption preferences might enhance BGP growth. Second, even if the sign of $dg^D/d\xi_2$ equals the sign of $\partial g^D/\partial \hat{m}^{c/C}$, the economic interpretation of the result is not correct, because it ignores the influence of ξ_2 via the $\hat{m}^{a/A}$- and the $|\hat{\varepsilon}^{u_c,c}|$-channels.

4.2 The Pure Relative Consumption Case

In this subsection we consider three specifications of pure relative consumption preferences in which relative wealth does not matter at all for utility. From Proposition 4 we know that in the absence of the relative wealth motive the common growth rate g^D is independent of the strength of the relative consumption motive. We show that two of the three specifications seem to yield contradictory results if the analysis is carried out by means of the standard approach.

Specification #1: Both in the status and in the habit persistence literature, the following specification is widely used:

$$V(c, H) = \frac{1}{1-\theta} \left[\left(c/H^{\beta} \right)^{1-\theta} - 1 \right], \qquad 0 < \beta < 1, \tag{53}$$

where H denotes the household's reference level. The simplest version of (53) results from the assumption that the reference level is given by average consumption in the economy, $H = C$. The resulting version of (53), $V(c, C)$, corresponds to the following specification of pure relative consumption preferences:[8]

$$u(c, c/C) = \frac{1}{1-\theta}\left\{\left[c^{1-\beta}(c/C)^{\beta}\right]^{1-\theta} - 1\right\}. \tag{54}$$

According to (54), the representative household derives utility from a geometric weighted average of absolute and relative consumption. In case of $\beta = 0$, relative consumption is irrelevant so that only absolute consumption matters. The greater the parameter β, the more important is relative consumption as compared to absolute consumption. Note that changes in the parameter β affect not only the exponent of relative consumption c/C, but also the exponent of absolute consumption c given by $1-\beta$.

Obviously, (54) is a special case of our specification (44) that is obtained by setting $\xi_2 = \beta$, $\xi_1 = 1 - \xi_2 = 1 - \beta$, and $\xi_3 = 0$. Hence, according to (47), (48), and (49) we have

$$\hat{m}^{c/C} = \frac{\beta}{1-\beta}, \qquad \hat{m}^{a/A} = 0, \qquad |\hat{\varepsilon}^{u_c,c}| = 1 + (\theta - 1)(1-\beta),$$

$$\hat{\sigma} = \frac{1}{1 + (\theta-1)(1-\beta)}, \qquad \hat{\eta} = 0, \qquad g^D = \frac{f_k(1, L) - \rho}{1 + (\theta - 1)(1-\beta)}.$$

From the solution of the decentralized growth rate it follows that g^D depends positively (negatively) on β if $\theta > 1$ ($\theta < 1$). Employing the standard approach and restricting the analysis to the mere calculation of $dg^D/d\beta$, one could come to the conclusion that the specification of u given by (54) allows for the possibility that the decentralized BGP growth rate is affected by the strength of the relative consumption motive although relative wealth is irrelevant for utility. Even more surprisingly, one could conclude that relative consumption preferences enhance GDP growth. Obviously, these two misleading conclusions of the standard approach are at variance with part ii) of Proposition 4 given in this paper.

From the considerations made above, the fallacy of the traditional approach is already clear: it overlooks the fact that changes in the parameter β affect two fundamental factors, namely $\hat{m}^{c/C}$ and $|\hat{\varepsilon}^{u_c,c}|$. More precisely, an increase in the exponent of relative consumption given by $\xi_2 = \beta$ is inevitably associated with a decrease in the exponent of absolute consumption given by $\xi_1 = 1 - \beta$. Both the rise in $\xi_2 = \beta$ and the fall in $\xi_1 = 1 - \beta$ cause the percentage-MRS of relative consumption $\hat{m}^{c/C} = \xi_2/\xi_1$ to increase. However, this rise in $\hat{m}^{c/C}$ does not affect at all the comparison-induced extra return factor, since $\hat{\eta} = \hat{m}^{a/A}/\left(1 + \hat{m}^{c/C}\right) = 0$ holds due to the assumption that relative wealth is irrelevant for utility ($\xi_3 = 0 \Rightarrow \hat{m}^{a/A} = \xi_3/\xi_1 = 0$). In addition, the fall in $\xi_1 = 1 - \beta$ leads to an ambiguous reaction of the effective

[8] Specifications that are equivalent to (54) are, for instance, used in Harbaugh (1996) [Equation (1)], Grossmann (1998) [Equation (7)], Fisher and Hof (2000) [Equation (20)], Liu and Turnovsky (2005) [Equation (14b)], García-Peñalosa and Turnovsky (2008) [Equation (22)], and Nakamoto (2009) [Equation (22)]. In other models that employ (53), H is treated as predetermined stock variable that evolves over time. Carroll et al. (1997) and Alvarez-Cuadrado et al. (2004) distinguish in this context between outward- and inward-looking agents. The case of outward-looking households (external habits) is modeled by assuming that $\dot{H} = \gamma(C - H)$, which, in turn, implies that the reference stock H is calculated as an exponentially declining weighted average of past *average* levels of consumption in the economy, $H(t) = \gamma \int_{-\infty}^{t} e^{\gamma(\tau - t)} C(\tau)\, d\tau$. The case of inward-looking households (internal habits) is obtained by setting $\dot{H} = \gamma(c - H)$ so that the household's reference stock H is calculated as an exponentially declining weighted average of her *own* past levels of consumption: $H(t) = \gamma \int_{-\infty}^{t} e^{\gamma(\tau - t)} c(\tau)\, d\tau$. Koyuncu and Turnovsky (2010) restrict their attention to external habits. Chen (2007) considers only internal habits, but uses a more complicated differential equation for H.

elasticity of intertemporal substitution $\hat{\sigma} = 1/\left|\hat{\varepsilon}^{u_c,c}\right| = 1/\left[1+(\theta-1)\,\xi_1\right]$, where $\text{sgn}(\partial\hat{\sigma}/\partial\beta) = -\text{sgn}(\partial\hat{\sigma}/\partial\xi_1) = \text{sgn}(\theta-1)$. For the case in which $\theta > 1$ holds, the following results can be easily derived (if $\theta < 1$, then the opposite results obtain): A rise in β causes $\hat{\sigma}$ to increase. According to (29), the rise in $\hat{\sigma}$ leads to an increase in the decentralized growth rate g^D. Proposition 2 and its interpretation made it clear that – in general – the rise in $\hat{\sigma}$ exerts both a direct and an indirect effect on g^D. However, since $\hat{\eta} = 0$, there is only the direct effect: The willingness of private households to substitute future absolute consumption for present absolute consumption increases. The resulting rise in the propensity to save stimulates the common rate of growth of aggregate capital, consumption, and output.

Our analysis made it clear that the reaction of the growth rate does not result from the change in the exponent of relative consumption, $\xi_2 = \beta$. Instead, it is caused by the change in the exponent of absolute consumption, $\xi_1 = 1-\beta$. If relative wealth is irrelevant for utility, then this change in $\xi_1 = 1-\beta$ affects the common growth rate only via its ambiguous effect on the effective elasticity of intertemporal substitution $\hat{\sigma} = 1/\left[1+(\theta-1)\,\xi_1\right]$. In other words, the change in the willingness to substitute relative consumption for absolute consumption (i.e., the change in $\hat{m}^{c/C}$) does not exert any effect. Only the change in the willingness to substitute absolute consumption intertemporally matters, where the sign of this effect depends on the sign of $\theta - 1$.

Specification #2: The specification of $V(c,C)$ that is employed by Galí (1994) is equivalent to the following representation:

$$V(c,C) = \frac{1}{1-\theta}c^{1-\theta}C^{\gamma\theta}, \qquad \theta > 0, \quad \theta \neq 1, \quad \gamma < 1.$$

In the context of the pure relative consumption approach, this representation of $V(c,C)$ is obtained under the assumption that u takes the following form:

$$u(c,c/C) = \frac{1}{1-\theta}\left[c^{[1-(1-\gamma)\theta]/(1-\theta)}\,(c/C)^{-\gamma\theta/(1-\theta)}\right]^{1-\theta}. \tag{55}$$

Obviously, (55) is obtained by setting $\xi_2 = -\gamma\theta/(1-\theta)$, $\xi_1 = 1-\xi_2$, and $\xi_3 = 0$ in the representation of the instantaneous utility function given by (44) and ignoring the constant term "-1". Please note that in contrast to the specification #1 given by (54), the exponents of both absolute consumption and relative consumption depend on the parameter θ of the CRRA function. It is obvious that $V(c,C)$ is well-behaved in the sense that $V_c > 0$ and $V_{cc} < 0$. To ensure that $u = u(c,c/C)$ satisfies all assumptions made in (2), $u_c > 0$, $u_{cc} < 0$, and $u_{c/C} > 0$, we have to introduce appropriate restrictions on the parameter γ for any given value of θ. In Appendix C.2, we show that these restrictions depend on whether $\theta > 1$ or $\theta < 1$ holds:

$$\text{Case A:} \quad \theta > 1, \quad 0 < \gamma < \frac{\theta-1}{\theta}, \qquad \text{Case B:} \quad \theta < 1, \quad -\frac{1-\theta}{\theta} < \gamma < 0. \tag{56}$$

Using (47), (48), and (49), it can be shown that $\hat{\varepsilon}^{u_c,c} = -(1-\gamma)\,\theta < 0$, $\hat{m}^{a/A} = 0$, $\hat{\eta} = 0$,

$$\hat{m}^{c/C} = \frac{-\gamma\theta}{1-(1-\gamma)\,\theta} > 0, \qquad \hat{\sigma} = \frac{1}{(1-\gamma)\,\theta} > 0, \qquad g^D = \frac{f_k(1,L)-\rho}{(1-\gamma)\,\theta} > 0.$$

It is verified at first glance that the decentralized growth rate g^D depends positively on the parameter γ. Hence, restricting the analysis to the mere calculation of the partial derivative $\partial g^D/\partial\gamma > 0$, one could draw the erroneous conclusion that g^D depends on the strength of the relative consumption motive, although relative wealth is irrelevant for utility. Meanwhile the reader is familiar with the reason for this potential fallacy. Changes in γ affect not only ξ_2, but also $\xi_1 = 1 - \xi_2$, and, hence, the effective elasticity of intertemporal substitution $\hat{\sigma} = 1/[1 + (\theta - 1)\xi_1] = 1/[(1-\gamma)\theta]$. A rise in γ causes g^D to rise unambiguously. The only reason for this positive growth effect is that the willingness to substitute absolute consumption intertemporally depends positively on γ. By contrast, the change in $\hat{m}^{c/C}$ that results from the rise in γ is irrelevant, since due to the absence of the relative wealth motive, we have $\hat{m}^{a/A} = 0$ and, hence, $\hat{\eta} = 0$, i.e., the comparison-induced extra return factor is identical to zero, irrespective of the strength of the relative consumption motive as measured by the percentage-MRS $\hat{m}^{c/C}$.

Specification #3: One of the illustrations employed by Liu and Turnovsky (2005) [see equation (14a), p. 1110] is equivalent to

$$V(c, C) = \frac{1}{1-\theta}\left\{\left[\left(\frac{c^{\varphi} - \kappa C^{\varphi}}{1-\kappa}\right)^{1/\varphi}\right]^{1-\theta} - 1\right\}, \qquad 0 < \kappa < 1, \quad 0 < 1 - \varphi < \theta. \tag{57}$$

Considering the limiting case in which $\varphi \to 1$ and ignoring the irrelevant expression $1 - \kappa$ in the denominator, we get the specification that would obtain in Ljungqvist and Uhlig (2000) [see p. 357] if – in contrast to the authors' assumption – work effort were not treated as endogenously determined but as exogenously given: $V(c, C) = (1-\theta)^{-1}\left[(c - \kappa C)^{1-\theta} - 1\right]$. In terms of the pure relative consumption approach, the more general Liu and Turnovsky (2005) version (57) corresponds to the following specification of the instantaneous utility function:[9]

$$u(c, c/C) = \frac{1}{1-\theta}\left\{\left[c\left(\frac{1 - \kappa(c/C)^{-\varphi}}{1-\kappa}\right)^{1/\varphi}\right]^{1-\theta} - 1\right\}. \tag{58}$$

It is easily verified that (58) is obtained by setting

$$\xi_1 = 1, \qquad Q(c/C, a/A) = \left(\frac{1 - \kappa(c/C)^{-\varphi}}{1-\kappa}\right)^{1/\varphi}$$

in our own *general* representation of the instantaneous utility function given by (38). Taking into account that $\xi_1 = 1$, $\varepsilon^{Q,c/C}(c/C, a/A) = \kappa(c/C)^{-\varphi} / \left[1 - \kappa(c/C)^{-\varphi}\right]$, and $\varepsilon^{Q,a/A}(c/C, a/A) = 0$, it then follows from (41), (42), and (43) that $\hat{\varepsilon}^{u_c,c} = -\theta$, $\hat{m}^{a/A} = 0$, $\hat{\eta} = 0$,

$$\hat{m}^{c/C} = \frac{\kappa}{1-\kappa}, \qquad \hat{\sigma} = \frac{1}{\theta}, \qquad g^D = \frac{f_k(1, L) - \rho}{\theta}.$$

Obviously, the common growth rate g^D depends neither on κ nor on φ, i.e., $\partial g^D/\partial\kappa =$

[9] The assumptions $0 < \kappa < 1$ and $0 < 1 - \varphi < \theta$ ensure that both $u = u(c, c/C)$ and $V = V(c, C)$ are well-behaved in the sense that all assumptions made in (2) and (4), respectively, are satisfied (for a proof see Appendix C.3).

$\partial g^D/\partial\varphi = 0$. Hence, in this specification, the standard approach does not run the risk of making erroneous conclusions regardless of whether it uses $\partial g^D/\partial\kappa$ or $\partial g^D/\partial\varphi$ to assess the implications of relative consumption preferences in the absence of the relative wealth motive. The properties of g^D result from the following facts: First, since $\xi_1 = 1$ holds by assumption, the effective elasticity of intertemporal substitution, $\hat{\sigma} = 1/|\hat{\varepsilon}^{u_c,c}| = 1/\theta$, is independent of both κ and φ. Second, the positive dependence of $\hat{m}^{c/C} = \kappa/(1-\kappa)$ on κ is irrelevant for g^D because, due to the absence of the relative wealth motive, we have $\hat{m}^{a/A} = 0$ and, hence, the comparison-induced extra return factor $\hat{\eta}$ is identical to zero, irrespective of the magnitude of $\hat{m}^{c/C}$.

4.3 Specifications in which both relative consumption and relative wealth matter for utility

In this subsection we consider three specifications in which both relative consumption and relative wealth matter for utility.

Specification #4: Setting

$$\xi_1 = 1, \qquad Q(c/C, a/A) = [\Omega(c/C)]^{\gamma} [\Psi(a/A)]^{\delta}, \tag{59}$$

in our general representation of the instantaneous utility function given by (38), where $\Omega(\cdot)$ and $\Psi(\cdot)$ denote functions, we obtain a specification that is equivalent to the utility function used by Tournemaine and Tsoukis (2008) for the case of an *exogenously* given labor supply. Our presentation differs only with respect to the notation. It is easily verified that $\hat{\varepsilon}^{Q,c/C} \equiv \varepsilon^{Q,c/C}(1,1) = \gamma\varepsilon^{\Omega,c/C}(1)$ and $\hat{\varepsilon}^{Q,a/A} \equiv \varepsilon^{Q,a/A}(1,1) = \delta\varepsilon^{\Psi,a/A}(1)$, where $\varepsilon^{\Omega,c/C}(1)$ denotes the elasticity of the function Ω with respect to c/C, evaluated at $c/C = 1$, while $\varepsilon^{\Psi,a/A}(1)$ denotes the elasticity of the function Ψ with respect to a/A, evaluated at $a/A = 1$. Substituting these two results and $\xi_1 = 1$ into (41) and (43), we obtain

$$\hat{m}^{c/C} = \gamma\varepsilon^{\Omega,c/C}(1), \qquad \hat{m}^{a/A} = \delta\varepsilon^{\Psi,a/A}(1), \qquad |\hat{\varepsilon}^{u_c,c}| = \theta, \tag{60}$$

$$g^D = \frac{f_k(1,L) - \rho + \dfrac{\delta\varepsilon^{\Psi,a/A}(1)}{1+\gamma\varepsilon^{\Omega,c/C}(1)} \times f(1,L)}{\theta + \dfrac{\delta\varepsilon^{\Psi,a/A}(1)}{1+\gamma\varepsilon^{\Omega,c/C}(1)}}. \tag{61}$$

Employing a solution for g^D that is equivalent to (61), Tournemaine and Tsoukis (2008) show the following (see Proposition 1 given on p. 315): i) without the relative consumption motive ($\gamma = 0$), growth increases in the strength of the relative wealth motive, ii) without the relative wealth motive ($\delta = 0$), economic growth remains unaffected by the strength of the relative consumption motive, iii) if both relative consumption and relative wealth matter for utility, a stronger relative consumption motive unambiguously harms economic growth.

These assertions coincide with the results of our fundamental factor approach summarized in our own Proposition 4. The reason for this fact can be explained as follows: Since Tournemaine and Tsoukis (2008) set $\xi_1 = 1$ and, in addition, use the implicit assumption that the two expressions $\gamma\varepsilon^{\Omega,c/C}(1)$ and $\delta\varepsilon^{\Psi,a/A}(1)$ are independent of each other, there is a one-to-one correspondence between the three fundamental factors $|\hat{\varepsilon}^{u_c,c}|$, $\hat{m}^{c/C}$, and $\hat{m}^{a/A}$, and the three parameters θ, $\gamma\varepsilon^{\Omega,c/C}(1)$, and $\delta\varepsilon^{\Psi,a/A}(1)$. Changes in $\gamma\varepsilon^{\Omega,c/C}(1)$ affect only the strength of the relative consumption motive, while changes in $\delta\varepsilon^{\Psi,a/A}(1)$ alter solely the intensity of the relative wealth motive. Hence, it is legitimate to assess the implications of the relative consumption [resp. relative wealth] preferences by applying the standard approach, i.e., by analyzing the dependence of g^D on $\gamma\varepsilon^{\Omega,c/C}(1)$ [resp. $\delta\varepsilon^{\Psi,a/A}(1)$]. The following properties of g^D are easily verified: (a) $\partial g^D/\partial\left[\delta\varepsilon^{\Psi,a/A}(1)\right] > 0$ holds regardless of whether $\gamma\varepsilon^{\Omega,c/C}(1) = 0$ or $\gamma\varepsilon^{\Omega,c/C}(1) > 0$. (b) If $\delta\varepsilon^{\Psi,a/A}(1) = 0$, then (61) simplifies to $g^D = (1/\theta)\left[f_k(1,L) - \rho\right]$ so that g^D is independent of $\gamma\varepsilon^{\Omega,c/C}(1)$. (c) If $\delta\varepsilon^{\Psi,a/A}(1) > 0$, then $\partial g^D/\partial\left[\gamma\varepsilon^{\Omega,c/C}(1)\right] < 0$. These mathematical results confirm the results of Tournemaine and Tsoukis (2008).

While the specification used by Tournemaine and Tsoukis (2008) yields correct results, it does not encompass specifications in which instantaneous utility depends on a geometric weighted average of absolute consumption, relative consumption, and relative wealth. Hence, it covers neither the specifications (54) and (55) that were used for the illustrations #1 and #2, nor the two specifications that we analyze in the remainder of this subsection. In the following, we assume that the instantaneous utility function is of the simple form given by (44). We analyze two alternative cases in which $\xi_1 + \xi_2 + \xi_3 = 1$ holds in addition to $\xi_1 > 0$, $\xi_2 > 0$, and $\xi_3 > 0$.

Specification #5: We assume that the two exponents of relative consumption and relative wealth, $\xi_2 > 0$ and $\xi_3 > 0$, are independent of each other and satisfy the condition that $\xi_2 + \xi_3 < 1$. From the constraint $\xi_1 + \xi_2 + \xi_3 = 1$ it then follows that the exponent of absolute consumption is automatically given by $\xi_1 = 1 - \xi_2 - \xi_3 > 0$ so that

$$u(c, c/C, a/A) = \frac{1}{1-\theta}\left\{\left[c^{1-\xi_2-\xi_3}(c/C)^{\xi_2}(a/A)^{\xi_3}\right]^{1-\theta} - 1\right\}. \tag{62}$$

Substitution of $\xi_1 = 1 - \xi_2 - \xi_3$ into (47), (48), and (49) yields

$$\hat{m}^{c/C} = \frac{\xi_2}{1-\xi_2-\xi_3}, \quad \hat{m}^{a/A} = \frac{\xi_3}{1-\xi_2-\xi_3}, \quad |\hat{\varepsilon}^{u_c,c}| = 1 + (\theta-1)(1-\xi_2-\xi_3)$$

$$\hat{\sigma} = \frac{1}{1+(\theta-1)(1-\xi_2-\xi_3)}, \qquad \hat{\eta} = \frac{\xi_3}{1-\xi_3},$$

$$g^D = \frac{f_k(1,L) - \rho + \dfrac{\xi_3}{1-\xi_3} f(1,L)}{1 + (\theta-1)(1-\xi_2-\xi_3) + \dfrac{\xi_3}{1-\xi_3}}. \tag{63}$$

From Proposition 4 (see part ii) we know that if relative wealth matters for utility so that $\hat{m}^{a/A} > 0$, then g^D depends negatively on the strength of the relative consumption motive as measured by the percentage-MRS $\hat{m}^{c/C}$. The standard analysis would question the validity of this result (that coincides with an assertion made in Proposition 1 of Tournemaine and Tsoukis (2008)) by using (63) and pointing out that i) $\partial g^D/\partial\xi_2 > 0$ holds for $\theta > 1$, and ii) $\partial g^D/\partial\xi_2 = 0$ holds for $\theta = 1$. However, once again the criticism would be based on the erroneous presumption that a change in ξ_2 represents a ceteris paribus change in the strength of the relative consumption motive. Our fundamental factor approach elucidates the main problem of the standard approach: Since changes in the exponent of relative consumption ξ_2 affect the two fundamental factors $\hat{m}^{c/C}$ and $\hat{m}^{a/A}$, and, in addition, the third fundamental factor $|\hat{\varepsilon}^{u_c,c}|$ provided that $\theta \neq 1$, the partial derivative $\partial g^D/\partial\xi_2$ yields misleading information with respect to the growth effects of relative consumption preferences. More precisely, under the specification (62), any increase in the exponent of relative consumption ξ_2 is necessarily associated with a fall in the exponent of absolute consumption $\xi_1 = 1 - \xi_2 - \xi_3$ of equal magnitude, while ξ_3 remains unchanged. This fall in ξ_1 reinforces the rise in the percentage-MRS of relative consumption $\hat{m}^{c/C} = \xi_2/\xi_1$ due to the decrease in the denominator. In contrast to specifications #1 and #2, the fall in ξ_1 also leads to an increase in the percentage-MRS of relative wealth $\hat{m}^{a/A} = \xi_3/\xi_1$. Hence, there is also an increase in the strength of the relative wealth motive. The increase in $\hat{m}^{a/A}$ causes the comparison-induced extra return factor $\hat{\eta}$ to rise, while the rise in $\hat{m}^{c/C}$ exerts a negative effect on $\hat{\eta}$. Note that the two effects offset each other perfectly. This follows from the fact that $\hat{\eta} = \xi_3/(1-\xi_3)$ is independent of ξ_2. Hence, although relative wealth matters for utility, changes in ξ_2 affect the decentralized growth rate g^D, if at all, only via the resulting change in the effective elasticity of intertemporal substitution, i.e., via the $\hat{\sigma}$-channel. There is no effect via the $\hat{\eta}$-channel. The fall in $\xi_1 = 1 - \xi_2 - \xi_3$ that results from an increase in ξ_2 leads to an ambiguous reaction of $|\hat{\varepsilon}^{u_c,c}|$ and $\hat{\sigma} = 1/|\hat{\varepsilon}^{u_c,c}|$, where $\text{sgn}(\partial\hat{\sigma}/\partial\xi_2) = -\text{sgn}(\partial\hat{\sigma}/\partial\xi_1) = \text{sgn}(\theta - 1)$. If $\theta > 1$ holds, then the increase in ξ_2 causes $\hat{\sigma}$ to rise. According to (29) the rise in $\hat{\sigma}$ causes the decentralized growth rate g^D to increase. Analogously, if $\theta < 1$, then the opposite results obtain. Finally, if $\theta = 1$, then changes in ξ_2 affect neither $\hat{\eta}$ nor $\hat{\sigma}$ so that g^D is independent of ξ_2.

Specification #6: Finally, we use a specification in which status is not only implicitly, but also explicitly taken into account. More precisely, we assume that the instantaneous utility function u can be written as $u(c, c/C, a/A) \equiv \tilde{u}(c, s(c/C, a/A))$, where s stands for status. To ensure that $u(c, c/C, a/A)$ is of the simple form given by (44) and that $\xi_1 + \xi_2 + \xi_3 = 1$ holds, we employ the following specifications of $\tilde{u}(c, s)$ and $s(c/C, a/A)$:

$$\tilde{u}(c, s) = \frac{1}{1-\theta}\left[\left(c^{1-\beta}s^{\beta}\right)^{1-\theta} - 1\right], \qquad s(c/C, a/A) = (c/C)^{\phi}(a/A)^{1-\phi}, \tag{64}$$

where $\theta > 0$, $0 < \beta < 1$, $1 + (\theta - 1)(1-\beta) > 0$, and $0 < \phi < 1$. The following properties of the function $\tilde{u}(c, s)$ are easily verified:

$$m^s(c, s) \equiv \frac{s}{c} \times \frac{\tilde{u}_s(c, s)}{\tilde{u}_c(c, s)} = \frac{\beta}{1-\beta} \equiv \hat{m}^s > 0, \quad \forall (c, s) \in \Theta_{\tilde{u}}, \tag{65}$$

$$\varepsilon^{\tilde{u}_c,c}(c, s) \equiv \frac{c\tilde{u}_{cc}(c, s)}{\tilde{u}_c(c, s)} = -[1 + (\theta - 1)(1-\beta)] \equiv \hat{\varepsilon}^{\tilde{u}_c,c} < 0, \quad \forall (c, s) \in \Theta_{\tilde{u}}, \tag{66}$$

where $\Theta_{\tilde{u}}$ denotes the domain of $\tilde{u}(c, s)$. Please note that m^s is the main innovation of the new specification. It represents the *percentage-MRS* of status s for absolute consumption c. Hence, m^s is the appropriate measure of the intensity of the quest for overall status that is determined by both relative consumption and relative wealth. The term $\varepsilon^{\tilde{u}_c,c}$ denotes the elasticity of the marginal utility of absolute consumption $\tilde{u}_c(c, s)$ with respect to absolute consumption. The status function $s(c/C, a/A)$ exhibits the property that its elasticities with respect to relative consumption and relative wealth, respectively, are constant functions over its domain Θ_s so that

$$\varepsilon^{s,c/C}(c/C, a/A) = \phi \equiv \hat{\varepsilon}^{s,c/C} > 0, \quad \varepsilon^{s,a/A}(c/C, a/A) = 1 - \phi \equiv \hat{\varepsilon}^{s,a/A} > 0$$

hold for all $(c/C, a/A) \in \Theta_s$. The specification

$$u(c, c/C, a/A) = \frac{1}{1-\theta}\left\{\left[c^{1-\beta}(c/C)^{\beta\phi}(a/A)^{\beta(1-\phi)}\right]^{1-\theta} - 1\right\} \tag{67}$$

that results from (64) seems to be the natural extension of the pure relative consumption specification #1 [see (54)] to the case in which also relative wealth matters. It is obvious from (67) that

$$\xi_1 = 1 - \beta > 0, \qquad \xi_2 = \beta\phi > 0, \qquad \xi_3 = \beta(1-\phi) > 0 \tag{68}$$

so that $\xi_1 + \xi_2 + \xi_3 = 1$. It is easily verified that

$$\hat{m}^{c/C} = \hat{m}^s \times \hat{\varepsilon}^{s,c/C} = \frac{\beta}{1-\beta} \times \phi > 0, \quad \hat{m}^{a/A} = \hat{m}^s \times \hat{\varepsilon}^{s,a/A} = \frac{\beta}{1-\beta} \times (1-\phi) > 0. \tag{69}$$

$$\left|\hat{\varepsilon}^{u_c,c}\right| = \left|\hat{\varepsilon}^{\tilde{u}_c,c}\right| = 1 + (\theta - 1)(1-\beta). \tag{70}$$

The corresponding common growth rate g^D is obtained by substituting (68) into (49). According to (69), the percentage-MRS of x for absolute consumption c, $\hat{m}^x$, where $x = c/C$ and a/A, can be written as the product of the percentage-MRS of status s for absolute consumption c, $\hat{m}^s$, and the elasticity of status s with respect to x, $\hat{\varepsilon}^{s,x}$. Hence, variations in $\hat{m}^s$ cause both $\hat{m}^{c/C}$ and $\hat{m}^{a/A}$ to change, but leave the percentage-MRS of relative wealth a/A for relative consumption c/C, $m^{a/A}/m^{c/C}$, unaffected. From (65), (66), (69), and (70) it is obvious that the simplicity of the specification (67) entails two significant drawbacks with respect to the application of the standard analysis: i) Since changes in the parameter ϕ affect both $\hat{m}^{c/C}$ and $\hat{m}^{a/A}$ (via the changes in both $\hat{\varepsilon}^{s,c/C}$ and $\hat{\varepsilon}^{s,a/A}$), the partial derivative $\partial g^D/\partial\phi$ is unsuited to analyze the effects of ceteris paribus changes in the intensity of the relative consumption motive or the relative wealth motive. ii) The partial derivative $\partial g^D/\partial\beta$ is inappropriate to analyze the effects of a change in the intensity of the quest for overall status. This is due to the following fact: If $\theta \neq 1$, then a change in β affects not only the willingness to substitute status for absolute consumption as measured by $\hat{m}^s$, but also the willingness to substitute absolute consumption intertemporally as determined by $1/\left|\hat{\varepsilon}^{\tilde{u}_c,c}\right| = 1/\left|\hat{\varepsilon}^{u_c,c}\right|$.

It can be shown that the standard analysis allows for correct results if i) the specification (64) is replaced by

$$\tilde{u}(c, s) = \frac{1}{1-\theta}\left[\left(c^{\chi_1}s^{\chi_2}\right)^{1-\theta} - 1\right], \qquad s(c/C, a/A) = (c/C)^{\phi_1}(a/A)^{\phi_2},$$

and ii) any functional dependence between the parameters θ, χ_1, χ_2, ϕ_1, and ϕ_2 is ruled out by assumption. For a thorough analysis of this alternative, more general specification see Appendix C.4.

5 The socially planned solution – General results and potential fallacies of the standard analysis

It is well known that in the model with standard preferences i) the growth rate of the decentralized economy is inefficiently low due to the knowledge spillovers and ii) the social optimum can be replicated by optimally granting subsidies on capital or production. To avoid other distortions, these subsidies have to be financed with a lump-sum tax [see, for instance, Barro and Sala-i-Martin (1995), p. 146–152]. In the following we analyze how this result is modified by the introduction of relative consumption and relative wealth preferences.

The benevolent social planner internalizes not only the knowledge spillovers in the production sector but also the externalities that result from relative consumption and relative wealth preferences. Since by assumption households are identical in every respect, the social planner will assign identical consumption paths to the individual households so that $c = C$ holds. This, in turn, implies that the resulting time paths of individual wealth are also identical so that $a = A = K$. Using these aspects, the optimization problem of the social planner can be reduced to the following simple problem: Maximize overall utility of the representative household as given by $\int_0^\infty e^{-\rho t} u\left(C, 1, 1\right) dt$, subject to the economy's resource constraint $\dot{K} = f\left(1, L\right) K - C$ and the initial condition $K\left(0\right) = K_0$ by choosing the time path of aggregate (= average) consumption C optimally. The current-value Hamiltonian of this optimization problem is given by $H = u\left(C, 1, 1\right) + \mu\left[f\left(1, L\right) K - C\right]$, where the costate variable μ denotes the shadow price of capital. The necessary optimality conditions for an interior equilibrium, $H_C = 0$ and $\dot{\mu} = \rho\mu - H_K$, can be written as

$$\mu = u_c\left(C, 1, 1\right), \tag{71}$$

$$\dot{\mu} = -\left[f\left(1, L\right) - \rho\right]\mu. \tag{72}$$

If, in addition, the transversality condition given by

$$\lim_{t\to\infty} e^{-\rho t}\mu K = 0 \tag{73}$$

holds, then the necessary optimality conditions are also sufficient, where this property follows from the fact that $u_{cc} < 0$ holds by assumption. Taking the time derivative of the first-order condition (71) and plugging the result into (72) yields the Euler equation of aggregate consumption in the socially planned economy as

$$\frac{\dot{C}}{C} = \sigma^P\left(C\right)\left[f\left(1, L\right) - \rho\right], \qquad \sigma^P\left(C\right) \equiv -\frac{1}{\varepsilon^{u_c, c}\left(C, 1, 1\right)}, \tag{74}$$

where $\sigma^P\left(C\right)$ is the elasticity of intertemporal substitution in the socially planned economy (the superscript "P" stands for "Planner"). Please note that neither the percentage-MRS of relative consumption $m^{c/C}$ nor the percentage-MRS of relative wealth $m^{a/A}$ are present in the Euler equation. Consequently, in the socially planned economy there is no counterpart $\eta^P\left(C\right)$ to the comparison-induced extra return factor $\eta^D\left(C\right)$ that plays a crucial role in the decentralized economy. For the results discussed below it is important that according to (24) the social marginal product of capital exceeds the private marginal product, $f(1,L) > f_k(1,L)$.

Since $u_c > 0$ holds by assumption, it follows from (71) that $\mu(t) > 0$ for $t \geq 0$. Hence, integration of (72) yields the result that the transversality condition (73) is equivalent to

$$\lim_{t\to\infty} e^{-f(1,L)t} K(t) = 0. \tag{75}$$

Similar to the decentralized market economy, we restrict our attention to the case in which the preferences exhibit the property that a BGP exists.

Proposition 6. *(The socially optimal BGP – existence)*

A) If the instantaneous utility function $u = u(c, c/C, a/A)$ satisfies the conditions given in (32) [Proposition 3],

$$m^{c/C}(C,1,1) = \hat{m}^{c/C}, \quad m^{a/A}(C,1,1) = \hat{m}^{a/A}, \quad \varepsilon^{u_c,c}(C,1,1) = \hat{\varepsilon}^{u_c,c}, \quad \forall C > 0,$$

where $\hat{m}^{c/C} \geq 0$, $\hat{m}^{a/A} \geq 0$ (with $\max\{\hat{m}^{c/C}, \hat{m}^{a/A}\} > 0$), and $\hat{\varepsilon}^{u_c,c} < 0$, then the effective elasticities of intertemporal substitution in both the socially planned economy and the decentralized economy are constant functions of C, where the corresponding constant levels are identical:

$$\sigma^P(C) = \sigma^D(C) = \frac{1}{|\hat{\varepsilon}^{u_c,c}|} \equiv \hat{\sigma} \qquad \forall C > 0. \tag{76}$$

If, in addition, the condition

$$\max\left\{\frac{(\hat{\sigma}-1) f(1,L)}{\hat{\sigma}}, 0\right\} < \rho < f(1,L) \tag{77}$$

is satisfied, then an economically meaningful BGP exists in the socially planned economy. Along the BGP, the constant common growth rate of consumption and capital $g^P = (\dot{C}/C)^P = (\dot{K}/K)^P$ and the constant consumption-capital ratio $(C/K)^P$ are given by

$$g^P = \hat{\sigma}[f(1,L) - \rho] > 0, \qquad (C/K)^P = (1-\hat{\sigma}) f(1,L) + \hat{\sigma}\rho > 0. \tag{78}$$

B) The model has no transitional dynamics.

For a proof of proposition 6 see Appendix D.1. Next, we compare the decentralized solution with the socially planned one. The aggregate resource constraints are identical in the two economies: $\dot{K} = f(1,L)K - C$. In contrast to this common differential equation for aggregate capital, the Euler equations for aggregate consumption that are given by

$$\frac{\dot{C}}{C} = \hat{\sigma}\left[f_k(1,L) + \hat{\eta} \times \frac{C}{K} - \rho\right], \qquad \text{and} \qquad \frac{\dot{C}}{C} = \hat{\sigma}[f(1,L) - \rho],$$

differ. According to Proposition 6, assumption (32) implies that the effective elasticities of intertemporal substitution of the decentralized and the socially planned economy are identical, $\sigma^P(C) = \sigma^D(C) = \hat{\sigma}$, $\forall C > 0$ [see (76)]. Hence, it is verified at first glance that the decentralized growth rate g^D deviates from its socially optimal counterpart g^P if and only if the decentralized effective rate of return $f_k(1, L) + \hat{\eta} \times (C/K)^D$ deviates from the social marginal product of capital $f(1, L)$. If the comparison-induced extra return factor equals zero, $\hat{\eta} = 0$, then the decentralized effective rate of return simplifies to the private marginal product of capital $f_k(1, L)$. Taking into account that $f_k(1, L) < f(1, L)$ [see (24)] holds because individual firms do not internalize knowledge spillovers, we obtain that $g^D < g^P$ holds for $\hat{\eta} = 0$. Hence, if relative wealth does not matter for utility so that $\hat{m}^{a/A} = 0$ and $\hat{\eta} = \hat{m}^{a/A} / \left(1 + \hat{m}^{c/C}\right) = 0$, then the growth rate in the decentralized economy is inefficiently low. In the following we analyze the dependence of the socially optimal growth rate g^P and the difference of the growth rates $g^P - g^D$ (henceforth, the growth rate gap) on $\hat{\eta}$ and $\hat{\sigma}$ and the two fundamental factors $\hat{m}^{a/A}$ and $\hat{m}^{c/C}$ in detail.

In the following proposition and corollary we assume that i) the instantaneous utility function u satisfies the conditions (32) given in Proposition 3, and ii) the subjective discount rate ρ satisfies the conditions given by (26) and (77) so that in both the decentralized economy and the socially planned economy an economically meaningful BGP exists.

Proposition 7. *(The dependence of g^P and $g^P - g^D$ on $\hat{\eta}$ and $\hat{\sigma}$)*

i) The growth rate gap $g^P - g^D$ decreases monotonically as the comparison-induced extra return factor $\hat{\eta}$ increases, because the decentralized growth rate g^D depends positively on $\hat{\eta}$, while the socially optimal growth rate g^P is independent of $\hat{\eta}$. There exists a threshold $\hat{\eta}^{crit} > 0$ with the following properties: If $0 \leq \hat{\eta} < \hat{\eta}^{crit}$, then $g^P - g^D > 0$, i.e., the decentralized growth rate is inefficiently low. If $\hat{\eta} = \hat{\eta}^{crit}$, then $g^P - g^D = 0$ and the decentralized growth rate equals its socially optimal counterpart. Finally, if $\hat{\eta} > \hat{\eta}^{crit}$, then $g^P - g^D < 0$ so that decentralized growth is too high as compared to the social optimum.

$$\frac{\partial g^P}{\partial \hat{\eta}} = 0, \qquad \frac{\partial \left(g^P - g^D\right)}{\partial \hat{\eta}} = -\frac{\partial g^D}{\partial \hat{\eta}} < 0,$$

$$sgn\left(g^P - g^D\right) = sgn\left(\hat{\eta}^{crit} - \hat{\eta}\right), \quad \textit{where } \hat{\eta}^{crit} \equiv \frac{f(1, L) - f_k(1, L)}{(1 - \hat{\sigma}) f(1, L) + \hat{\sigma}\rho} > 0.$$

ii) An increase in the effective elasticity of intertemporal substitution $\hat{\sigma}$ causes both g^P and g^D to rise. A sufficient (but not necessary) condition for a positive dependence of $g^P - g^D$ on $\hat{\sigma}$ is that the decentralized growth rate is inefficiently low, $g^P > g^D$:

$$\frac{\partial g^P}{\partial \hat{\sigma}} > 0, \quad \frac{\partial g^D}{\partial \hat{\sigma}} > 0, \quad \frac{\partial \left(g^P - g^D\right)}{\partial \hat{\sigma}} = \frac{1}{\hat{\sigma}} \left(g^P - \frac{g^D}{1 + \hat{\sigma}\hat{\eta}}\right) \geq \frac{1}{\hat{\sigma}} \left(g^P - g^D\right).$$

For a proof of Proposition 7 see Appendix D.2. An important aspect of the proof of item i) is the fact that the growth rate gap can be written in the form $g^P - g^D = \Lambda \times \left(\hat{\eta}^{\text{crit}} - \hat{\eta}\right)$, where $\Lambda > 0$. In the following corollary we address the dependence of g^P and $g^P - g^D$ on the fundamental factors $\hat{m}^{a/A}$ and $\hat{m}^{c/C}$.

Corollary 3. *(The dependence of g^P and $g^P - g^D$ on $\hat{m}^{a/A}$ and $\hat{m}^{c/C}$)*

i) The gap $g^P - g^D$ decreases monotonically as $\hat{m}^{a/A}$ increases because the decentralized growth rate g^D depends positively on $\hat{m}^{a/A}$, while the socially optimal growth rate g^P is *independent of $\hat{m}^{a/A}$. There exists a threshold $\left(\hat{m}^{a/A}\right)^{crit} > 0$ with the following properties: If $0 \leq \hat{m}^{a/A} < \left(\hat{m}^{a/A}\right)^{crit}$, then $g^P - g^D > 0$, i.e., the decentralized growth rate is inefficiently low. If $\hat{m}^{a/A} = \left(\hat{m}^{a/A}\right)^{crit}$, then $g^P - g^D = 0$ and the decentralized growth rate equals its socially optimal counterpart. Finally, if $\hat{m}^{a/A} > \left(\hat{m}^{a/A}\right)^{crit}$, then $g^P - g^D < 0$ so that decentralized growth is too high as compared to the social optimum. The length of the interval $[0, \left(\hat{m}^{a/A}\right)^{crit})$, in which the decentralized growth rate is inefficiently low, depends positively on $\hat{m}^{c/C}$.*

$$\frac{\partial g^P}{\partial \hat{m}^{a/A}} = 0, \qquad \frac{\partial \left(g^P - g^D\right)}{\partial \hat{m}^{a/A}} = -\frac{\partial g^D}{\partial \hat{m}^{a/A}} < 0,$$

$$sgn\left(g^P - g^D\right) = sgn\left[(\hat{m}^{a/A})^{crit} - \hat{m}^{a/A}\right],$$

$$where \ \left(\hat{m}^{a/A}\right)^{crit} \equiv \frac{\left[f(1,L) - f_k(1,L)\right]\left(1 + \hat{m}^{c/C}\right)}{\left[1 - (1/\left|\hat{\varepsilon}^{u_c,c}\right|)\right] f(1,L) + (1/\left|\hat{\varepsilon}^{u_c,c}\right|)\rho} > 0.$$

ii) If relative wealth matters for utility so that $\hat{m}^{a/A} > 0$, then $g^P - g^D$ depends positively on $\hat{m}^{c/C}$ because a rise in $\hat{m}^{c/C}$ causes g^D to decrease, while g^P remains unchanged. However, if $\hat{m}^{a/A} = 0$, i.e., in the absence of the relative wealth motive, $g^P - g^D$ is independent of $\hat{m}^{c/C}$, since changes in $\hat{m}^{c/C}$ affect neither g^P nor g^D.

$$\frac{\partial g^P}{\partial \hat{m}^{c/C}} = 0, \qquad sgn\left[\frac{\partial \left(g^P - g^D\right)}{\partial \hat{m}^{c/C}}\right] = -sgn\left(\frac{\partial g^D}{\partial \hat{m}^{c/C}}\right) = sgn\left(\hat{m}^{a/A}\right).$$

The mathematical results given in Corollary 3 are easily obtained by 1) using the fact that $\hat{\eta} = \hat{m}^{a/A}/\left(1 + \hat{m}^{c/C}\right)$ and $\hat{\sigma} = 1/\left|\hat{\varepsilon}^{u_c,c}\right|$, 2) taking into account that $\partial g^P/\partial \hat{\eta} = 0$ holds according to Proposition 7, and 3) recalling that $\partial g^D/\partial \hat{m}^{a/A} > 0$ and $\text{sgn}\left(\partial g^D/\partial \hat{m}^{c/C}\right) = -\text{sgn}\left(\hat{m}^{a/A}\right)$ hold according to Proposition 4. The positive dependence of $(\hat{m}^{a/A})^{\text{crit}}$ on $\hat{m}^{c/C}$ is verified at first glance. All other statements made in Corollary 3 follow directly from the mathematical assertions.

These results make clear that neither relative consumption preferences nor relative wealth preferences affect the socially optimal growth rate g^P. In addition, if relative wealth is irrelevant for utility, then the decentralized growth rate g^D i) is lower than its socially optimal counterpart due to the capital externality in the production function, $g^D < g^P$, and ii) is independent of the strength of the relative consumption motive that is correctly measured by the percentage-MRS $\hat{m}^{c/C}$. Hence, in the absence of relative wealth preferences, the gap $g^P - g^D$ is independent of $\hat{m}^{c/C}$, too. Users of the standard approach who employ the specification #1 given by (54), $u(c, c/C) = (1-\theta)^{-1}\left\{[c^{1-\beta}(c/C)^{\beta}]^{1-\theta} - 1\right\}$, and misinterpret the parameter β as the correct measure of the strength of the relative consumption motive might question our assertions by using the following results: The parameter setting $\xi_2 = \beta$, $\xi_1 = 1 - \xi_2 = 1 - \beta$, and $\xi_3 = 0$ implies that $\hat{\sigma} = [1 + (\theta - 1)\xi_1]^{-1} = [1 + (\theta - 1)(1 - \beta)]^{-1}$ and

$$g^P = \frac{f(1,L) - \rho}{1 + (\theta - 1)(1 - \beta)}, \qquad g^D = \frac{f_k(1,L) - \rho}{1 + (\theta - 1)(1 - \beta)},$$

$$g^P - g^D > 0, \qquad \frac{\partial \left(g^P - g^D\right)}{\partial \beta} = \frac{(\theta - 1)\left[f(1, L) - f_k(1, L)\right]}{\left[1 + (\theta - 1)(1 - \beta)\right]^2}.$$

It is obvious that $\text{sgn}[\partial \left(g^P - g^D\right)/\partial\beta] = \text{sgn}(\partial\hat{\sigma}/\partial\beta) = \text{sgn}(\theta - 1)$. If $\theta > 1$, then a rise in β causes both g^D and g^P to increase, where the rise in g^P exceeds that of g^D so that $g^P - g^D$ increases. Analogously, if $\theta < 1$, then a rise in β causes both g^D and g^P to decrease, where the fall in g^P exceeds that of g^D so that the gap $g^P - g^D$ decreases but remains strictly positive. From our analysis it is obvious that these reactions result exclusively from the change in the exponent of *absolute* consumption, $\xi_1 = 1 - \beta$, and the associated change in $\hat{\sigma}$. By contrast, the change in the exponent of *relative* consumption, $\xi_2 = \beta$, affects neither g^D nor g^P. In other words, changes in β cause g^D, g^P, and $g^P - g^D$ to react because the change in $\xi_1 = 1 - \beta$ alters the common effective elasticity of intertemporal substitution, $\sigma^D = \sigma^P = \hat{\sigma}$, and hence the willingness of the representative household to substitute absolute consumption over time in both the decentralized and the socially planned economy. By contrast, the strength of the relative consumption motive does not play any role.

Our result that the socially optimal growth rate g^P is independent of both relative consumption and relative wealth preferences might be also questioned erroneously by using the standard approach and employing specifications #3, #5, and #6. Our fundamental factor approach shows, however, that also in these instances changes in the parameters that seem to measure the strength of the relative consumption motive and/or the relative wealth motive actually affect g^P only via the associated change in the exponent of absolute consumption ξ_1 and the resulting reaction of the effective elasticity of intertemporal substitution, $\sigma^P = \hat{\sigma}$.

6 Conclusions

We use a novel approach for analyzing the effects of relative consumption and relative wealth preferences in the context of an otherwise standard AK-model with homogeneous agents and exogenous labor supply. We put special emphasis on the identification of the fundamental factors that ultimately determine both the decentralized and the socially optimal long-run growth rates. Our approach allows to analyze *separately* the effects of changes in i) the strength of the relative consumption motive, ii) the strength of the relative wealth motive, and iii) households' willingness to substitute absolute consumption intertemporally on the long-run growth rate by considering *ceteris paribus changes* in the corresponding fundamental factor. We show that there are specifications of the instantaneous utility function in which such ceteris paribus thought experiments cannot be carried out in the context of the standard approach that does not identify the fundamental factors. Instead, the standard approach restricts attention to the dependence of the growth rate on the parameters of the instantaneous utility function that *seem* to determine the strength of the relative consumption and relative wealth motives. Using our fundamental factor approach, we show that in a widely used type of utility function a parameter that seems to affect only the strength of the relative consumption motive actually also influences both the strength of the relative wealth motive and the willingness to substitute absolute consumption intertemporally. Since the standard approach is unaware of the latter two effects, it attributes the total growth effect that results from a change in this parameter erroneously to the change in the strength of the relative consumption motive. It is thus possible that the resulting assertions of the standard approach with respect to the implications of relative consumption preferences on long-run growth are not only quantitatively, but also qualitatively flawed. These erroneous conclusions apply to both the decentralized and the socially planned economy. We also provide specifications of the utility function in which the standard analysis yields correct results.

To obtain correct results and to explain the pitfalls of the standard analysis, we draw heavily on the Euler equation for aggregate consumption. The introduction of relative consumption and relative wealth preferences affects the Euler equation of the decentralized economy by modifying the elasticity of intertemporal substitution and the rate of return of wealth accumulation. In principle, the *effective* elasticity of intertemporal substitution may depend on the strength of the relative consumption motive. This possibility vanishes, however, in case that we introduce weak restrictions on the utility function that are sufficient for the existence of a balanced growth path (BGP). Consequently, in the presence of these restrictions all effects can be explained by means of the *effective* rate of return that is defined as the sum of the market rate of return and an extra return that results from social comparisons based on both relative wealth and relative consumption. The assumption that labor supply is exogenously given implies that relative consumption and relative wealth preferences do not affect the market rate of return. Hence, these preferences influence the long-run growth rate only via their effect on the comparison-induced extra return (CIER). Using this fact we derive the following results: i) The CIER depends positively on the strength of the relative wealth motive, irrespective of the strength of the relative consumption motive. Consequently, the willingness to save and the implied BGP growth rate always depend positively on the strength of the relative wealth motive. ii) If relative wealth matters for utility, then a rise in the strength of the relative consumption motive causes the CIER to fall. In this situation the willingness to save decreases because the fall in relative consumption that is associated with a reduction in absolute consumption now leads to a greater decrease in instantaneous utility. Consequently, in the presence of relative wealth preferences the growth rate depends negatively on the strength of the relative consumption motive. iii) In the absence of the relative wealth motive the CIER is equal to zero irrespective of the strength of the relative consumption motive. Thus, in this case the strength of the relative consumption motive does not affect long-run growth.

We also show that iv) neither relative consumption nor relative wealth preferences affect the socially planned solution. Using the results given by i) – iv) we finally derive the following insights: v) In the absence of the relative wealth motive, the decentralized growth rate is inefficiently low, with the positive gap between the socially optimal growth rate and its decentralized counterpart being unaffected by the strength of the relative consumption motive. vi) If relative wealth matters for utility, then a rise in the strength of the relative consumption motive causes the growth rate gap to increase. The growth rate gap decreases, however, as the strength of the relative wealth motive increases. There is a critical level of the strength of the relative wealth motive such that the decentralized growth rate equals the socially optimal one. We give several illustrations in which the standard analysis might yield the erroneous conclusion that our results given in iii), iv), and v) are not robust with respect to the specification of the instantaneous utility function.

With respect to further research, we identify the following promising areas. Since the erroneous conclusions of the standard analysis are usually drawn in models with exogenous labor supply, we restricted our analysis to this case. This assumption simplified the presentation significantly because it has the additional advantage that the BGP growth rates can be calculated explicitly. Endogenizing the labor supply decision would lead to additional insights because it allows for the possibility that relative consumption and relative wealth preferences affect not only the comparison-induced extra return, but also the market rate of return. In this setting it would be particularly interesting to analyze the growth effects within a Romer (1990) framework in which technological progress is driven by purposeful R&D investments, which, in turn require labor in the form of scientists to produce new blueprints. Another interesting extension is to apply the fundamental factor approach to a framework with heterogeneous agents that, in addition, also allows for alternative assumptions with respect to the reference levels of consumption and wealth on which the comparisons of the agents are based.

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Appendix

A The model (Section 2)

A.1 The expressions for V_{cc} and $V_{cc}V_{aa} - (V_{ca})^2$

The definition $V(c, C, a, A) \equiv u(c, c/C, a/A)$ given in (3) implies that

$$\begin{aligned} V_{cc} &= u_{cc} + 2C^{-1}u_{c(c/C)} + C^{-2}u_{(c/C)(c/C)} \qquad (A.1) \\ V_{cc}V_{aa} - (V_{ca})^2 &= A^{-2}\left[u_{cc} + 2C^{-1}u_{c(c/C)} + C^{-2}u_{(c/C)(c/C)}\right]u_{(a/A)(a/A)} \\ &\quad -A^{-2}\left[u_{c(a/A)} + C^{-1}u_{(c/C)(a/A)}\right]^2 \qquad (A.2) \end{aligned}$$

Proof: The validity of (A.1) and (A.2) is easily verified by using the following results:

$$\begin{aligned} V_c &= u_c + C^{-1}u_{c/C}, \\ V_{cc} &= u_{cc} + 2C^{-1}u_{c(c/C)} + C^{-2}u_{(c/C)(c/C)}, \\ V_{ca} &= A^{-1}\left[u_{c(a/A)} + C^{-1}u_{(c/C)(a/A)}\right], \\ V_a &= A^{-1}u_{(a/A)}, \\ V_{aa} &= A^{-2}u_{(a/A)(a/A)}. \quad \blacksquare \end{aligned}$$

A.2 Properties of the production function

By assumption each firm $i \in [0, 1]$ employs the same technology so that $y_i = f(k_i, Bl_i)$ for $i \in [0, 1]$. Since by assumption the common production function f exhibits constant returns to scale the following equations hold for $i \in [0, 1]$ (all results are well-known from intermediary microeconomics):

$$y_i = f(k_i, Bl_i) = k_i f(1, Bl_i/k_i), \qquad (A.3)$$

$$f_k(k_i, Bl_i) = f_k(1, Bl_i/k_i), \qquad f_{(Bl)}(k_i, Bl_i) = f_{(Bl)}(1, Bl_i/k_i), \qquad (A.4)$$

$$f(k_i, Bl_i) = f_k(k_i, Bl_i)k_i + f_{(Bl)}(k_i, Bl_i)Bl_i. \qquad (A.5)$$

The equations given in (A.4) follow from the fact that the marginal products of capital f_k and effective labor labor $f_{(Bl)}$ are homogeneous of degree zero. Equation (A.5) results from the Euler theorem.

Real profits of firm $i \in [0, 1]$ denoted by π_i are given by $\pi_i = f(k_i, Bl_i) - rk_i - wl_i$. It can be verified at first glance that the necessary optimality conditions are given by

$$r = f_k(k_i, Bl_i), \qquad w = f_{(Bl)}(k_i, Bl_i)B, \qquad i \in [0, 1]. \qquad (A.6)$$

From (A.6) it is obvious that the first-order conditions of the representative firm can be written in the form given by (14):

$$r = f_k(k, Bl), \qquad w = f_{(Bl)}(k, Bl)B.$$

B The decentralized solution – Part I (Section 3)

B.1 Derivation of (15)

Using (A.4) and taking into account that by assumption $B = K$ holds, the necessary optimality conditions (A.6),

$$r = f_k(k_i, Bl_i), \qquad w = f_{(Bl)}(k_i, Bl_i)\, B, \qquad i \in [0,1],$$

can be rewritten as

$$r = f_k(1, Kl_i/k_i), \qquad w = f_{(Bl)}(1, Kl_i/k_i)\, K, \qquad i \in [0,1]. \tag{B.1}$$

The equations given in (B.1) imply that in a macroeconomic equilibrium each firm will choose the same capital-labor ratio.[10] It is easily verified that

$$k_i/l_i = K/L, \qquad i \in [0,1], \tag{B.2}$$

where K and L denote both the aggregate and the average values of capital and labor input, respectively. Substituting (B.2) into (B.1) we obtain the two equations given in (15):

$$r = f_k(1, L), \qquad w = f_{(Bl)}(1, L)\, K. \quad \blacksquare$$

B.2 Derivation of (17)

Substitution of $B = K$ and (B.2) into (A.3) yields $y_i = k_i f(1, L)$, for $i \in [0,1]$. This, in turn, implies that aggregate output Y in a macroeconomic equilibrium is given by

$$Y = f(1, L)\, K. \tag{B.3}$$

Using the Euler theorem (A.5) and the necessary optimality conditions (A.6) we obtain $y_i = rk_i + wl_i$. Since the adding-up theorem holds at the level of the individual firm, it holds at the aggregate level, too:

$$Y = rK + wL. \tag{B.4}$$

Combining (B.3) and (B.4) we obtain

$$rK + wL = Y = f(1, L)\, K. \tag{B.5}$$

Substitution of (16), i.e., $c = C$, $a = A = K$, and $l = L$ into the flow budget constraint (1), $\dot{a} = ra + wl - c$, yields

$$\dot{K} = rK + wL - C. \tag{B.6}$$

[10] In contrast to the common capital labor ratio, firms need not necessarily choose identical levels of capital input and labor input [see, for instance, the thorough analysis in Sargent (1987), pp. 7–10].

From (B.5) and (B.6) it then follows that

$$\dot{K} = Y - C = f(1, L) K - C. \tag{B.7}$$

Dividing both sides of (B.7) by K, we obtain (17):

$$\dot{K}/K = f(1, L) - C/K. \quad \blacksquare$$

B.3 Derivation of (24)

Using (A.3)–(A.5), the following equation is easily derived:

$$f(1, Bl_i/k_i) = f_k(1, Bl_i/k_i) + f_{(Bl)}(1, Bl_i/k_i)(Bl_i/k_i). \tag{B.8}$$

Substitution of (B.2) and $B = K$ into (B.8) yields

$$f(1, L) = f_k(1, L) + f_{(Bl)}(1, L) L.$$

Taking into account that $f_{(Bl)} > 0$, we obtain (24):

$$f(1, L) > f_k(1, L). \quad \blacksquare$$

B.4 Proof of Proposition 1

Proof of A) Assumption (25),

$$\sigma^D(C) = \hat{\sigma}, \qquad \eta^D(C) = \hat{\eta}, \qquad \forall C > 0, \tag{B.9}$$

where $\hat{\sigma} > 0$ and $\hat{\eta} \geq 0$ are constants, implies that the Euler equation for aggregate consumption (20) simplifies to

$$\dot{C}/C = \hat{\sigma}\left[f_k(1, L) + \hat{\eta} \times (C/K) - \rho\right]. \tag{B.10}$$

The differential equation for aggregate capital given by (17) is unaffected by the assumptions made in (25) [= (B.9)]. It is still given by

$$\dot{K}/K = f(1, L) - (C/K). \tag{B.11}$$

Taking into account that, by assumption, L is exogenously given and constant over time and that both $\hat{\sigma}$ and $\hat{\eta}$ are constants, it is obvious from (B.10) and (B.11) that there exists a balanced growth path (BGP) in which C and K grow at the same constant rate so that C/K remains unchanged over time. The steady-state value of the common growth rate of aggregate consumption and aggregate physical capital denoted by $g^D = (\dot{C}/C)^D = (\dot{K}/K)^D$ and the steady value of the consumption-capital ratio denoted by $(C/K)^D$ are determined by the following system of equations:

$$
\begin{aligned}
g^D &= \hat{\sigma}\left[f_k(1,L) + \hat{\eta} \times (C/K)^D - \rho\right], \\
g^D &= f(1,L) - (C/K)^D.
\end{aligned}
$$

Solving this system of two equations for g^D and $(C/K)^D$, we obtain

$$g^D = \frac{f_k(1,L) - \rho + \hat{\eta} f(1,L)}{(1/\hat{\sigma}) + \hat{\eta}}, \tag{B.12}$$

$$(C/K)^D = \frac{(1/\hat{\sigma}) f(1,L) - [f_k(1,L) - \rho]}{(1/\hat{\sigma}) + \hat{\eta}}. \tag{B.13}$$

From (B.12) it is obvious that

$$g^D > 0 \Leftrightarrow \rho < f_k(1,L) + \hat{\eta} f(1,L) \equiv \rho^g. \tag{B.14}$$

From (B.13) it follows that

$$(C/K)^D > 0 \Leftrightarrow \rho > f_k(1,L) - (1/\hat{\sigma}) f(1,L) \equiv \rho^{C/K}. \tag{B.15}$$

Since, by assumption, $\eta^D(C) = \hat{\eta} \geq 0$ holds for $\forall C > 0$, the transversality condition (23) simplifies to

$$\lim_{t \to \infty} \exp\left\{-\int_0^t \left[f_k(1,L) + \hat{\eta} \times \frac{C(v)}{K(v)}\right] dv\right\} K(t) = 0.$$

Along the BGP we have $C/K = (C/K)^D$ and $\dot{K}/K = g^D$ at any point in time. Hence, the transversality condition requires that $-[f_k(1,L) + \hat{\eta} \times (C/K)^D] + g^D < 0$. Using the fact that

$$-\left[f_k(1,L) + \hat{\eta} \times (C/K)^D\right] + g^D = -\frac{[(1/\hat{\sigma}) - 1][f_k(1,L) + \hat{\eta} f(1,L)] + (1+\hat{\eta})\rho}{(1/\hat{\sigma}) + \hat{\eta}},$$

we obtain

$$-\left[f_k(1,L) + \hat{\eta} \times (C/K)^D\right] + g^D < 0 \Leftrightarrow \rho > \frac{[1 - (1/\hat{\sigma})][f_k(1,L) + \hat{\eta} f(1,L)]}{1+\hat{\eta}} \equiv \rho^{TC}, \tag{B.16}$$

where the superscript "TC" stands for "transversality condition".

From (B.14) it is obvious that $\rho^g > 0$. By contrast, both $\rho^{C/K}$ and ρ^{TC} [see (B.15) and (B.16)] may be of either sign. It is easily verified that

$$\rho^{TC} - \rho^{C/K} = \frac{[(1/\hat{\sigma}) + \hat{\eta}][f(1,L) - f_k(1,L)]}{1+\hat{\eta}}.$$

Taking into account that $f(1,L) > f_k(1,L)$ [see (24)] it is clear that $\rho^{C/K} < \rho^{TC}$. Moreover, we have

$$\rho^g - \rho^{TC} = \frac{[(1/\hat{\sigma}) + \hat{\eta}][f_k(1,L) + \hat{\eta} f(1,L)]}{1+\hat{\eta}} > 0.$$

These results imply that $\rho^{C/K} < \rho^{TC} < \rho^g$ holds. Hence, if the condition $\rho^{TC} < \rho < \rho^g$ is satisfied (so that also $\rho^{C/K} < \rho$ holds), then the solutions given by (B.12) and (B.13) are economically meaningful in the sense that 1) the common growth rate g^D is strictly positive (due to $\rho < \rho^g$), 2) the consumption-capital ratio $(C/K)^D$ is strictly positive (due to $\rho > \rho^{C/K}$), and 3) the transversality condition (23) is satisfied (due to $\rho > \rho^{TC}$). By assumption, ρ is strictly positive. Hence, the condition $\rho^{TC} < \rho < \rho^g$ is satisfied if and only if either $\rho^{TC} < 0 < \rho < \rho^g$ or $0 < \rho^{TC} < \rho < \rho^g$ holds. These two conditions can be represented jointly in the following compact way:

$$\max\{\rho^{TC}, 0\} < \rho < \rho^g. \tag{B.17}$$

Substituting the definitions of ρ^{TC} and ρ^g given in (B.14) and (B.16) into (B.17) yields

$$\max\left\{\frac{[1-(1/\hat{\sigma})]\,[f_k(1,L)+\hat{\eta}f(1,L)]}{1+\hat{\eta}}, 0\right\} < \rho < f_k(1,L)+\hat{\eta}f(1,L). \tag{B.18}$$

Obviously, the condition (B.18) is identical to the condition (26) given in Proposition 1.

Above we have shown that if (B.18) [= (26)] holds, then $g^D > 0$ and $(C/K)^D > 0$. Combining these two results with (B.12) and (B.13), we obtain

$$g^D = \frac{f_k(1,L)-\rho+\hat{\eta}f(1,L)}{(1/\hat{\sigma})+\hat{\eta}} > 0, \tag{B.19}$$

$$(C/K)^D = \frac{(1/\hat{\sigma})\,f(1,L)-[f_k(1,L)-\rho]}{(1/\hat{\sigma})+\hat{\eta}} > 0. \tag{B.20}$$

The representations (B.19) and (B.20) are identical to the representations (27) and (28) given in Proposition 1. ■

Proof of B) Finally, we show that if the condition (25) [= (B.9)] is satisfied, then the model has no transitional dynamics. Let $Z \equiv C/K$. Since K is a state variable and C is a control variable, $Z = C/K$ is a control-like variable (this notion is used by Barro and Sala-i-Martin (1995) on p. 162). In contrast to K, both C and $Z = C/K$ can jump at any point in time. From (B.10), (B.11), and $C/K = Z$ it then follows that

$$\dot{C}/C = \hat{\sigma}\,[f_k(1,L)+\hat{\eta}\times Z-\rho], \tag{B.21}$$

$$\dot{K}/K = f(1,L)-Z, \tag{B.22}$$

which, in turn implies that

$$\begin{aligned}\dot{Z} &= [(\dot{C}/C)-(\dot{K}/K)]Z \\ &= -\{f(1,L)-\hat{\sigma}\,[f_k(1,L)-\rho]-(1+\hat{\sigma}\hat{\eta})\,Z\}\,Z \equiv \Phi(Z).\end{aligned}$$

Solving $\dot{Z} = \Phi(Z) = 0$ for Z, we obtain $\{Z = 0\}$ and $\{Z = Z^D\}$, where

$$Z^D = \frac{(1/\hat{\sigma})\,f(1,L)-[f_k(1,L)-\rho]}{(1/\hat{\sigma})+\hat{\eta}}. \tag{B.23}$$

Obviously, Z^D given by (B.23) is identical to $(C/K)^D$ given by (B.13) [= (28)]. If (B.18) [= (26)] holds, then $Z^D = (C/K)^D > 0$, so that Z^D is the economically meaningful steady state value of the consumption-capital ratio. From

$$\Phi'(Z) = (1+\hat{\sigma}\hat{\eta})\,Z - \{f(1,L) - \hat{\sigma}[f_k(1,L)-\rho] - (1+\hat{\sigma}\hat{\eta})\,Z\}$$

and (B.23) it follows that

$$\Phi'\left(Z^D\right) = (1+\hat{\sigma}\hat{\eta})\,Z^D > 0,$$

because the expression between curly brackets vanishes. $\Phi'\left(Z^D\right) > 0$ implies that the economically meaningful steady state of the differential equation $\dot{Z} = \Phi(Z)$ is unstable. Hence, the equilibrium path of Z has no transitional dynamics, i.e., $Z(t) = Z^D$ for $t \geq 0$. The initial value of the jump variable Z has to be chosen in such a way that $Z(0) = Z^D$. From $Z = C/K$ and $Z^D = (C/K)^D$ it then follows that the initial value of the jump variable C has to be chosen according to $C(0) = (C/K)^D \times K_0$, where $(C/K)^D$ is given by (B.13) [= (28)] and K_0 is exogenously given.

Since $Z(t) = Z^D$ holds for $t \geq 0$, it then follows from (B.21), (B.22), and (B.19) that

$$\begin{aligned}
\dot{C}/C &= \hat{\sigma}[f_k(1,L) + \hat{\eta}\times Z - \rho] = \frac{f_k(1,L) - \rho + \hat{\eta} f(1,L)}{(1/\hat{\sigma}) + \hat{\eta}} = g^D > 0,\\
\dot{K}/K &= f(1,L) - Z = \frac{f_k(1,L) - \rho + \hat{\eta} f(1,L)}{(1/\hat{\sigma}) + \hat{\eta}} = g^D > 0
\end{aligned}$$

hold for $t \geq 0$. The growth rates of consumption and capital are constant over time, identical and equal to g^D. Consequently, the growth rates of C and K have no transitional dynamics. ■

B.5 Proof of Proposition 2

We restrict our attention to a proof of the mathematical results presented in (29) and (30). Along the economically meaningful BGP the common growth rate of consumption and capital $g^D = (\dot{C}/C)^D = (\dot{K}/K)^D$ and the consumption-capital ratio $(C/K)^D$ have the following properties [see Proposition 1, (27) and (28)]:

$$\begin{aligned}
g^D &= \frac{f_k(1,L) - \rho + \hat{\eta} f(1,L)}{(1/\hat{\sigma}) + \hat{\eta}} > 0,\\
(C/K)^D &= \frac{(1/\hat{\sigma}) f(1,L) - [f_k(1,L) - \rho]}{(1/\hat{\sigma}) + \hat{\eta}} > 0.
\end{aligned}$$

Taking partial derivatives of g^D, $(C/K)^D$, and

$$\hat{\eta} \times (C/K)^D = \frac{\hat{\eta}\{(1/\hat{\sigma}) f(1,L) - [f_k(1,L) - \rho]\}}{(1/\hat{\sigma}) + \hat{\eta}}$$

with respect to $\hat{\sigma}$ and $\hat{\eta}$ we obtain

$$\frac{\partial g^D}{\partial \hat{\sigma}} = \frac{f_k(1,L) - \rho + \hat{\eta} f(1,L)}{\hat{\sigma}^2 [(1/\hat{\sigma}) + \hat{\eta}]^2} = \frac{g^D}{\hat{\sigma}^2 [(1/\hat{\sigma}) + \hat{\eta}]}, \tag{B.24}$$

$$\frac{\partial g^D}{\partial \hat{\eta}} = \frac{(1/\hat{\sigma}) f(1,L) - [f_k(1,L) - \rho]}{[(1/\hat{\sigma}) + \hat{\eta}]^2} = \frac{(C/K)^D}{(1/\hat{\sigma}) + \hat{\eta}}, \tag{B.25}$$

$$\frac{\partial (C/K)^D}{\partial \hat{\sigma}} = -\frac{f_k(1,L) - \rho + \hat{\eta} f(1,L)}{\hat{\sigma}^2 \left[(1/\hat{\sigma}) + \hat{\eta}\right]^2} = -\frac{g^D}{\hat{\sigma}^2 \left[(1/\hat{\sigma}) + \hat{\eta}\right]},$$

$$\frac{\partial (C/K)^D}{\partial \hat{\eta}} = -\frac{(1/\hat{\sigma}) f(1,L) - \left[f_k(1,L) - \rho\right]}{\left[(1/\hat{\sigma}) + \hat{\eta}\right]^2} = -\frac{(C/K)^D}{(1/\hat{\sigma}) + \hat{\eta}},$$

$$\frac{\partial \left[\hat{\eta} \times (C/K)^D\right]}{\partial \hat{\sigma}} = -\frac{\hat{\eta}\left[f_k(1,L) - \rho + \hat{\eta} f(1,L)\right]}{\hat{\sigma}^2 \left[(1/\hat{\sigma}) + \hat{\eta}\right]^2} = -\frac{\hat{\eta} g^D}{\hat{\sigma}^2 \left[(1/\hat{\sigma}) + \hat{\eta}\right]},$$

$$\frac{\partial \left[\hat{\eta} \times (C/K)^D\right]}{\partial \hat{\eta}} = \frac{(1/\hat{\sigma}) f(1,L) - \left[f_k(1,L) - \rho\right]}{\hat{\sigma}\left[(1/\hat{\sigma}) + \hat{\eta}\right]^2} = \frac{(C/K)^D}{\hat{\sigma}\left[(1/\hat{\sigma}) + \hat{\eta}\right]}.$$

Using these results and taking into account that $\hat{\sigma} > 0$ and $\hat{\eta} \geq 0$, we finally obtain (29) and (30):

$$\frac{\partial g^D}{\partial \hat{\sigma}} > 0, \quad \frac{\partial (C/K)^D}{\partial \hat{\sigma}} < 0, \quad \operatorname{sgn}\left(\frac{\partial \left[\hat{\eta} \times (C/K)^D\right]}{\partial \hat{\sigma}}\right) = -\operatorname{sgn}(\hat{\eta}),$$

$$\frac{\partial g^D}{\partial \hat{\eta}} > 0, \quad \frac{\partial (C/K)^D}{\partial \hat{\eta}} < 0, \quad \frac{\partial \left[\hat{\eta} \times (C/K)^D\right]}{\partial \hat{\eta}} > 0. \quad \blacksquare$$

B.6 Proof of Proposition 3

Proof of (33) In (32) we make the following assumptions:

$$m^{c/C}(C,1,1) = \hat{m}^{c/C}, \quad m^{a/A}(C,1,1) = \hat{m}^{a/A}, \quad \varepsilon^{u_c,c}(C,1,1) = \hat{\varepsilon}^{u_c,c}, \quad \forall C > 0,$$

where $\hat{m}^{c/C} \geq 0$, $\hat{m}^{a/A} \geq 0$ (with $\max\left\{\hat{m}^{c/C}, \hat{m}^{a/A}\right\} > 0$), and $\hat{\varepsilon}^{u_c,c} < 0$ are constants. From $m^{c/C}(C,1,1) = \hat{m}^{c/C}$, $\forall C > 0$, it then follows that $\varepsilon^{m^{c/C},c}(C,1,1) = 0$, $\forall C > 0$. Substituting the latter result and the assumptions made in (32) into the definitions of $\sigma^D(C)$ and $\eta^D(C)$ given by (21) and (22),

$$\sigma^D(C) \equiv -\left[\varepsilon^{u_c,c}(C,1,1) + \frac{m^{c/C}(C,1,1)}{1 + m^{c/C}(C,1,1)} \times \varepsilon^{m^{c/C},c}(C,1,1)\right]^{-1},$$

$$\eta^D(C) \equiv \frac{m^{a/A}(C,1,1)}{1 + m^{c/C}(C,1,1)},$$

we obtain (33),

$$\sigma^D(C) = \frac{1}{\left|\hat{\varepsilon}^{u_c,c}\right|} \equiv \hat{\sigma}, \qquad \eta^D(C) = \frac{\hat{m}^{a/A}}{1 + \hat{m}^{c/C}} \equiv \hat{\eta}, \qquad \forall C > 0. \ \blacksquare$$

Proof of (34) From Proposition 1 we know that if these results for $\hat{\sigma}$ and $\hat{\eta}$ satisfy condition (26),

$$\max\left\{\frac{\left[1 - (1/\hat{\sigma})\right]\left[f_k(1,L) + \hat{\eta} f(1,L)\right]}{1 + \hat{\eta}}, 0\right\} < \rho < f_k(1,L) + \hat{\eta} f(1,L),$$

then an economically meaningful decentralized BGP exists. In order to calculate the corresponding BGP growth we substitute the expressions for $\hat{\sigma}$ and $\hat{\eta}$ into Equation (27) given in

Proposition 1,

$$g^D = \frac{f_k(1,L) - \rho + \hat{\eta} f(1,L)}{(1/\hat{\sigma}) + \hat{\eta}}.$$

In doing so we finally obtain (34):

$$g^D = \frac{f_k(1,L) - \rho + \dfrac{\hat{m}^{a/A}}{1+\hat{m}^{c/C}} \times f(1,L)}{|\hat{\varepsilon}^{u_c,c}| + \dfrac{\hat{m}^{a/A}}{1+\hat{m}^{c/C}}}. \blacksquare$$

B.7 An extended version of Proposition 5

Proposition 8. *(Extended version of Proposition 5)*

Let the instantaneous utility function u result from the transformation T of a multiplicatively separable function v,

$$u(c, c/C, a/A) = T[v(c, c/C, a/A)], \qquad v(c, c/C, a/A) = P(c)\,Q(c/C, a/A). \tag{B.26}$$

A) i) If the functions $T(v)$, $P(c)$, and $Q(c/C, a/A)$ satisfy the conditions

$$T' > 0, \quad T'' < 0, \qquad P > 0, \quad P' > 0, \quad P'' \leq 0, \tag{B.27}$$

$$Q > 0, \quad Q_{c/C} \geq 0, \quad Q_{a/A} \geq 0, \quad Q_{c/C} > 0 \vee Q_{a/A} > 0, \tag{B.28}$$

over their corresponding domains, then the instantaneous utility function (B.26) is well-behaved in the sense that all assumptions made in (2) are satisfied, i.e., $u_c > 0$, $u_{cc} < 0$, $u_{c/C} \geq 0$, $u_{a/A} \geq 0$, and $u_{c/C} > 0 \vee u_{a/A} > 0$.

ii) The instantaneous utility function (B.26) is still well-behaved if in (B.27) the assumption $P'' \leq 0$ is replaced by a weaker assumption that does not rule out that $P'' > 0$ holds, namely

$$\varepsilon^{T',v}(P(c)\,Q(c/C, a/A))\,\varepsilon^{P,c}(c) + \varepsilon^{P',c}(c) < 0, \tag{B.29}$$

where $\varepsilon^{P,c}(c) \equiv P'(c) \times [P(c)]^{-1} c$ and $\varepsilon^{P',c}(c) \equiv P''(c) \times [P'(c)]^{-1} c$ denote the elasticities of $P(c)$ and $P'(c)$, respectively, with respect to c, while $\varepsilon^{T',v}(v) \equiv T''(v) \times [T'(v)]^{-1} v$ represents the elasticity of $T'(v)$ with respect to v. Consequently, $\varepsilon^{T',v}(P(c)\,Q(c/C, a/A))$ gives the value that $\varepsilon^{T',v}(v)$ takes at $v = P(c)\,Q(c/C, a/A)$.

iii) The assumptions $P > 0$ and $Q > 0$ ensure that $v > 0$ so that transformations T that are not defined for $v < 0$, such as functions of the CRRA-type, are not ruled out from the outset.

B) The instantaneous utility function (B.26), $u(c, c/C, a/A) = T[P(c)\,Q(c/C, a/A)]$, exhibits the property that

$$m^{c/C}(C,1,1) = \hat{m}^{c/C}, \quad m^{a/A}(C,1,1) = \hat{m}^{a/A}, \quad \eta^D(C) = \hat{\eta}, \quad \forall C > 0, \tag{B.30}$$

where $\hat{m}^{c/C}$, $\hat{m}^{a/A}$, and $\hat{\eta}$ are constants, holds if and only if the function $P(c)$ has the *form*

$$P(c) = \xi_0 c^{\xi_1}, \qquad \text{for } c > 0. \tag{B.31}$$

In order to ensure that $P(c) > 0$ and $P'(c) > 0$ hold as required by the sufficient conditions for the well-behavedness of the instantaneous utility function (B.26) given in A) it is assumed that

$$\xi_0 > 0, \qquad \xi_1 > 0. \tag{B.32}$$

C) *The instantaneous utility function*

$$u(c, c/C, a/A) = T\left[\xi_0 c^{\xi_1} Q(c/C, a/A)\right], \qquad \xi_0 > 0, \quad \xi_1 > 0 \tag{B.33}$$

that is obtained by substituting (B.31) into (B.26) and taking into account (B.32), has the property that

$$\varepsilon^{u_c,c}(C,1,1) = \hat{\varepsilon}^{u_c,c}, \quad \sigma^D(C) = \hat{\sigma}, \qquad \forall C > 0, \tag{B.34}$$

where $\hat{\varepsilon}^{u_c,c}$ and $\hat{\sigma}$ are constants, holds if and only if the function $T(v)$ has the form

$$T(v) = \kappa_0 + \kappa_1 \frac{v^{1-\theta} - 1}{1-\theta}, \qquad \text{for } v > 0. \tag{B.35}$$

In order to ensure that $T'(v) > 0$ and $T''(v) < 0$ hold as required by (B.27), and that also the condition (B.29) is satisfied, it is assumed that

$$\kappa_1 > 0, \quad \theta > 0, \quad 1 + (\theta - 1)\xi_1 > 0 \tag{B.36}$$

hold in addition to (B.32).

D) *The instantaneous utility function that is obtained by substituting (B.31) and (B.35) into (B.26), and taking into account (B.28), (B.32), and (B.36),*

$$u(c, c/C, a/A) = \kappa_0 + \frac{\kappa_1}{1-\theta}\left\{\left[\xi_0 c^{\xi_1} Q(c/C, a/A)\right]^{1-\theta} - 1\right\}, \tag{B.37}$$

where the parameters satisfy the conditions

$$\kappa_1 > 0, \quad \theta > 0, \quad \xi_0 > 0, \quad \xi_1 > 0, \quad 1 + (\theta - 1)\xi_1 > 0, \tag{B.38}$$

and the function $Q(c/C, a/A)$ satisfies the conditions given (B.28), has the following properties:

i) It is well-behaved in the sense that all assumptions made in (2) are satisfied.

ii) The corresponding alternative representation, $V(c, C, a, A) \equiv u(c, c/C, a/A)$, is well-behaved in the sense that all assumptions made in (4), i.e., $V_{cc} < 0$, and $V_{cc}V_{aa} - (V_{ca})^2 > 0$ if $u_{a/A} > 0$, are satisfied, if, in addition to (B.28) and (B.38), the conditions

$$0 < [1 + \xi_1(\theta - 1)]\xi_1 + \varepsilon^{Q,c/C}\left[2(\theta - 1)\xi_1 + \theta\varepsilon^{Q,c/C} - \varepsilon^{Q_{c/C},c/C}\right], \tag{B.39}$$

$$
\begin{aligned}
0 \;<\; & \left\{ \left[1+\xi_1(\theta-1)\right]\xi_1 + \varepsilon^{Q,c/C}\left[2(\theta-1)\xi_1 + \theta\varepsilon^{Q,c/C} - \varepsilon^{Q_{c/C},c/C}\right]\right\} \times \\
& \qquad\qquad\qquad \times\varepsilon^{Q,a/A}\left(\theta\varepsilon^{Q,a/A} - \varepsilon^{Q_{a/A},a/A}\right) \\
& -\left\{\left[(1-\theta)\xi_1 - \theta\varepsilon^{Q,c/C}\right]\varepsilon^{Q,a/A} + \varepsilon^{Q,c/C}\varepsilon^{Q_{c/C},a/A}\right\}^2
\end{aligned} \tag{B.40}
$$

are satisfied, where (B.40) is only relevant in case that relative wealth matters for utility so that $Q_{a/A} > 0$. *In (B.39) and (B.40) the following notation is used: The expressions* $\varepsilon^{Q,c/C} \equiv Q_{c/C} \times (c/C)\, Q^{-1}$ *and* $\varepsilon^{Q,a/A} \equiv Q_{a/A} \times (a/A)\, Q^{-1}$ *denote the elasticities of the function* $Q(c/C, a/A)$ *with respect to* c/C *and* a/A, *the expressions* $\varepsilon^{Q_{c/C},c/C} \equiv Q_{(c/C)(c/C)} \times (c/C)\left(Q_{c/C}\right)^{-1}$ *and* $\varepsilon^{Q_{c/C},a/A} \equiv Q_{(c/C)(a/A)} \times (a/A)\left(Q_{c/C}\right)^{-1}$ *are the elasticities of* $Q_{c/C}$ *with respect to* c/C *and* a/A, *while* $\varepsilon^{Q_{a/A},a/A} \equiv Q_{(a/A)(a/A)} \times (a/A)\left(Q_{a/A}\right)^{-1}$ *denotes the elasticity of* $Q_{a/A}$ *with respect to* a/A.

iii) The conditions given by (32) in Proposition 3 are satisfied, since $m^{c/C}(C,1,1) = \hat{m}^{c/C}$, $m^{a/A}(C,1,1) = \hat{m}^{a/A}$, *and* $\varepsilon^{u_c,c}(C,1,1) = \hat{\varepsilon}^{u_c,c}$ *holds for* $C > 0$, *where*

$$
\hat{m}^{c/C} \equiv \frac{\hat{\varepsilon}^{Q,c/C}}{\xi_1} \geq 0, \quad \hat{m}^{a/A} \equiv \frac{\hat{\varepsilon}^{Q,a/A}}{\xi_1} \geq 0, \quad \hat{\varepsilon}^{u_c,c} \equiv -\left[1+(\theta-1)\xi_1\right] < 0. \tag{B.41}
$$

In (B.41) the constants $\hat{\varepsilon}^{Q,c/C}$ *and* $\hat{\varepsilon}^{Q,a/A}$ *denote the values that the elasticities of the function* $Q(c/C, a/A)$ *with respect to* c/C *and* a/A, $\varepsilon^{Q,c/C}(c/C, a/A)$, *and* $\varepsilon^{Q,a/A}(c/C, a/A)$, *take in symmetric situations, i.e., at* $(c/C, a/A) = (1,1)$:

$$
\hat{\varepsilon}^{Q,c/C} \equiv \varepsilon^{Q,c/C}(1,1), \quad \hat{\varepsilon}^{Q,a/A} \equiv \varepsilon^{Q,a/A}(1,1). \tag{B.42}
$$

iv) The conditions given in (25) in Proposition 1 are satisfied, because $\sigma^D(C) = \hat{\sigma}$ *and* $\eta^D(C) = \hat{\eta}$ *hold for* $C > 0$, *where*

$$
\hat{\sigma} \equiv \frac{1}{1+(\theta-1)\xi_1} > 0, \qquad \hat{\eta} \equiv \frac{\hat{\varepsilon}^{Q,a/A}/\xi_1}{1+\hat{\varepsilon}^{Q,c/C}/\xi_1} \geq 0. \tag{B.43}
$$

If these constants $\hat{\sigma}$ *and* $\hat{\eta}$ *satisfy the condition (26),*

$$
\max\left\{\frac{\left[1-(1/\hat{\sigma})\right]\left[f_k(1,L)+\hat{\eta}f(1,L)\right]}{1+\hat{\eta}}, 0\right\} < \rho < f_k(1,L) + \hat{\eta}f(1,L),
$$

then according to Proposition 1 an economically meaningful decentralized BGP exists. The corresponding constant common growth rate is given by

$$
g^D = \frac{f_k(1,L) - \rho + \dfrac{\hat{\varepsilon}^{Q,a/A}/\xi_1}{1+\hat{\varepsilon}^{Q,c/C}/\xi_1} \times f(1,L)}{1+(\theta-1)\xi_1 + \dfrac{\hat{\varepsilon}^{Q,a/A}/\xi_1}{1+\hat{\varepsilon}^{Q,c/C}/\xi_1}}. \tag{B.44}
$$

v) The results for $\hat{m}^{c/C}$, $\hat{m}^{a/A}$, $\hat{\varepsilon}^{u_c,c}$, $\hat{\sigma}$, *and* $\hat{\eta}$ *given in (B.41), (B.43), and (B.44) are independent of the parameters* κ_0, κ_1, *and* ξ_0. *The well-behavedness of* $u(c, c/C, a/A)$ *given by (B.37) and its alternative representation* $V(c, C, a, A) \equiv u(c, c/C, a/A)$ *depends* *on the signs of ξ_0 and κ_1 [i.e., $\xi_0 > 0$ and $\kappa_1 > 0$ has to hold according to (B.32) and (B.36)], but not on the magnitudes of these two parameters. Hence, we can set, without loss of generality, $\kappa_0 = 0$, $\kappa_1 = 1$, and $\xi_0 = 1$, and employ the following representation of the utility function:*

$$u(c, c/C, a/A) = \frac{1}{1-\theta}\left\{\left[c^{\xi_1} Q(c/C, a/A)\right]^{1-\theta} - 1\right\}. \tag{B.45}$$

Proof

Preliminaries

The specification of the instantaneous utility function $u = u(c, c/C, a/A)$ given by (B.26),

$$u(c, c/C, a/A) = T[v(c, c/C, a/A)], \qquad v(c, c/C, a/A) = P(c) Q(c/C, a/A),$$

implies that

$$\begin{aligned}
u_c(c, c/C, a/A) &= T'[P(c) Q(c/C, a/A)] P'(c) Q(c/C, a/A), &\text{(B.46)}\\
u_{c/C}(c, c/C, a/A) &= T'[P(c) Q(c/C, a/A)] P(c) Q_{c/C}(c/C, a/A), &\text{(B.47)}\\
u_{a/A}(c, c/C, a/A) &= T'[P(c) Q(c/C, a/A)] P(c) Q_{a/A}(c/C, a/A), &\text{(B.48)}\\
u_{cc}(c, c/C, a/A) &= T''[P(c) Q(c/C, a/A)] \left[P'(c) Q(c/C, a/A)\right]^2 \\
&\quad + T'[P(c) Q(c/C, a/A)] P''(c) Q(c/C, a/A). &\text{(B.49)}
\end{aligned}$$

Equations (B.46)–(B.49) can be rewritten as

$$\begin{aligned}
u_c(c, c/C, a/A) &= c^{-1}[P(c) Q(c/C, a/A)] T'[P(c) Q(c/C, a/A)] \varepsilon^{P,c}(c), &\text{(B.50)}\\
u_{c/C}(c, c/C, a/A) &= (c/C)^{-1}[P(c) Q(c/C, a/A)] T'[P(c) Q(c/C, a/A)] \times \\
&\quad \times \varepsilon^{Q,c/C}(c/C, a/A), &\text{(B.51)}\\
u_{a/A}(c, c/C, a/A) &= (a/A)^{-1} P(c) Q(c/C, a/A) T'[P(c) Q(c/C, a/A)] \times \\
&\quad \times \varepsilon^{Q,a/A}(c/C, a/A), &\text{(B.52)}\\
u_{cc}(c, c/C, a/A) &= c^{-1} P'(c) Q(c/C, a/A) T'[P(c) Q(c/C, a/A)] \times \\
&\quad \times \left\{\varepsilon^{T',v}(P(c) Q(c/C, a/A)) \varepsilon^{P,c}(c) + \varepsilon^{P',c}(c)\right\}, &\text{(B.53)}
\end{aligned}$$

where

$$\begin{aligned}
\varepsilon^{P,c}(c) &\equiv P'(c) \times [P(c)]^{-1} c, &\text{(B.54)}\\
\varepsilon^{Q,c/C}(c/C, a/A) &\equiv Q_{c/C}(c/C, a/A) \times [Q(c/C, a/A)]^{-1}(c/C), &\text{(B.55)}\\
\varepsilon^{Q,a/A}(c/C, a/A) &\equiv Q_{a/A}(c/C, a/A) \times [Q(c/C, a/A)]^{-1}(a/A), &\text{(B.56)}\\
\varepsilon^{T',v}(v) &\equiv T''(v) \times \left[T'(v)\right]^{-1} v, &\text{(B.57)}\\
\varepsilon^{P',c}(c) &\equiv P''(c) \times \left[P'(c)\right]^{-1} c. &\text{(B.58)}
\end{aligned}$$

Here, $\varepsilon^{P,c}(c)$ and $\varepsilon^{P',c}(c)$ denote the elasticities of $P(c)$ and $P'(c)$, respectively, with respect to c, while $\varepsilon^{Q,c/C}(c/C, a/A)$ and $\varepsilon^{Q,a/A}(c/C, a/A)$ are the elasticities of the function $Q(c/C, a/A)$

with respect to c/C and a/A. The expression $\varepsilon^{T',v}(v)$ denotes the elasticity of $T'(v)$ with respect to v. Consequently, $\varepsilon^{T',v}(P(c)Q(c/C,a/A))$ gives the value that $\varepsilon^{T',v}(v)$ takes at $v = P(c)Q(c/C,a/A)$.

Proof of A)

Proof of A-i) From (B.46)–(B.49) is obvious that the assumptions given in (B.27) and (B.28),

$$T' > 0, \quad T'' < 0, \qquad P > 0, \quad P' > 0, \quad P'' \leq 0, \tag{B.59}$$

$$Q > 0, \quad Q_{c/C} \geq 0, \quad Q_{a/A} \geq 0, \quad Q_{c/C} > 0 \vee Q_{a/A} > 0, \tag{B.60}$$

ensure that the instantaneous utility function (B.26) is well-behaved in the sense that all assumptions made in (2) are satisfied, i.e., $u_c > 0$, $u_{cc} < 0$, $u_{c/C} \geq 0$, $u_{a/A} \geq 0$, and $u_{c/C} > 0 \vee u_{a/A} > 0$.

Proof of A-ii) From (B.53) it follows that if $P' > 0$, $Q > 0$, and $T' > 0$ holds, then $u_{cc}(c, c/C, a/A)$ is strictly negative if and only if

$$\varepsilon^{T',v}(P(c)Q(c/C,a/A))\,\varepsilon^{P,c}(c) + \varepsilon^{P',c}(c) < 0. \tag{B.61}$$

Consequently, the instantaneous utility function (B.26) is still well-behaved if in (B.27) the assumption $P'' \leq 0$ is replaced by a weaker assumption (B.61) that does not rule out that $P'' > 0$ holds.

Proof of A-iii) The assumptions $P(c) > 0$ and $Q(c/C,a/A) > 0$ that are included in (B.27) and (B.28) ensure that $v(c, c/C, a/A) = P(c)Q(c/C,a/A) > 0$. Consequently, in our approach transformations T that are not defined for $v < 0$, such as functions of the CRRA-type, are not ruled out from the outset. ■

Proof of B)

Substituting (B.50)–(B.52) into the definitions of $m^{c/C}(c, c/C, a/A)$ and $m^{a/A}(c, c/C, a/A)$ given in (11), we obtain

$$m^{c/C}(c, c/C, a/A) \equiv \frac{c/C}{c} \times \frac{u_{c/C}(c, c/C, a/A)}{u_c(c, c/C, a/A)} = \frac{\varepsilon^{Q,c/C}(c/C, a/A)}{\varepsilon^{P,c}(c)},$$

$$m^{a/A}(c, c/C, a/A) \equiv \frac{a/A}{c} \times \frac{u_{a/A}(c, c/C, a/A)}{u_c(c, c/C, a/A)} = \frac{\varepsilon^{Q,a/A}(c/C, a/A)}{\varepsilon^{P,c}(c)}.$$

Consequently, in symmetric situations, in which $c = C$ and $a = A$ hold, we have

$$m^{c/C}(C,1,1) = \hat{\varepsilon}^{Q,c/C}/\varepsilon^{P,c}(C), \tag{B.62}$$

$$m^{a/A}(C,1,1) = \hat{\varepsilon}^{Q,a/A}/\varepsilon^{P,c}(C), \tag{B.63}$$

where the constants $\hat{\varepsilon}^{Q,c/C}$ and $\hat{\varepsilon}^{Q,a/A}$ give the values that the elasticities $\varepsilon^{Q,c/C}$ and $\varepsilon^{Q,a/A}$ take at $(c/C, a/A) = (1,1)$:

$$\hat{\varepsilon}^{Q,c/C} \equiv \varepsilon^{Q,c/C}(1,1), \qquad \hat{\varepsilon}^{Q,a/A} \equiv \varepsilon^{Q,a/A}(1,1). \tag{B.64}$$

The assumptions made in (B.28), $Q > 0$, $Q_{c/C} \geq 0$, $Q_{a/A} \geq 0$, and $Q_{c/C} > 0 \vee Q_{a/A} > 0$, imply that

$$\hat{\varepsilon}^{Q,c/C} \geq 0, \qquad \hat{\varepsilon}^{Q,a/A} \geq 0, \qquad \hat{\varepsilon}^{Q,c/C} \vee \hat{\varepsilon}^{Q,a/A} > 0. \tag{B.65}$$

Substituting (B.62) and (B.63) into the definition of the comparison-induced extra return factor given by (22) we obtain

$$\eta^D(C) \equiv \frac{m^{a/A}(C,1,1)}{1+m^{c/C}(C,1,1)} = \frac{\hat{\varepsilon}^{Q,a/A}/\varepsilon^{P,c}(C)}{1+\hat{\varepsilon}^{Q,c/C}/\varepsilon^{P,c}(C)} = \frac{\hat{\varepsilon}^{Q,a/A}}{\varepsilon^{P,c}(C)+\hat{\varepsilon}^{Q,c/C}}. \tag{B.66}$$

From (B.62), (B.63), and (B.66) it is obvious that $m^{c/C}(C,1,1)$, $m^{a/A}(C,1,1)$, and $\eta^D(C)$ are constant functions of C if and only if the elasticity of the function $P(c)$ with respect to c, $\varepsilon^{P,c}(c)$, is a constant function of c. It is easily verified that $\varepsilon^{P,c}(c)$ is a constant function of c if and only if the function $P(c)$ has the form given by (B.31),

$$P(c) = \xi_0 c^{\xi_1}, \qquad c > 0, \tag{B.67}$$

where ξ_0 and ξ_1 are constants. These considerations prove the validity of the first assertion made in item B): The functions $m^{c/C}$, $m^{a/A}$, and η^D that result from the specification of the instantaneous utility function $u = u(c, c/C, a/A)$ given by (B.26) have the properties described in (B.30),

$$m^{c/C}(C,1,1) = \hat{m}^{c/C}, \quad m^{a/A}(C,1,1) = \hat{m}^{a/A}, \quad \eta^D(C) = \hat{\eta}, \quad \forall C > 0,$$

if and only if the function $P(c)$ has the form given by (B.67) [= (B.31)].

Next, we derive the parameter restrictions given in (B.32). In (B.27) it is assumed that both $P(c) > 0$ and $P'(c) > 0$ hold for $c > 0$. In order to ensure that (B.67) [= (B.31)] satisfies these two assumptions we have to introduce the following two restrictions with respect to its parameters:

$$\xi_0 > 0, \qquad \xi_1 > 0. \tag{B.68}$$

Obviously, the assumptions made in (B.68) coincide with those made in (B.32).

From (B.67) and (B.68) it follows that

$$\varepsilon^{P,c}(c) = \xi_1 > 0, \qquad \forall c > 0. \tag{B.69}$$

Using (B.69), (B.62), (B.63), (B.65), and (B.66) it is easily verified that

$$m^{c/C}(C,1,1) = \hat{\varepsilon}^{Q,c/C}/\xi_1 \equiv \hat{m}^{c/C} \geq 0, \qquad \forall C > 0, \tag{B.70}$$

$$m^{a/A}(C,1,1) = \hat{\varepsilon}^{Q,a/A}/\xi_1 \equiv \hat{m}^{a/A} \geq 0, \qquad \forall C > 0, \tag{B.71}$$

$$\eta^D(C) = \frac{\hat{\varepsilon}^{Q,a/A}/\xi_1}{1+\hat{\varepsilon}^{Q,c/C}/\xi_1} \equiv \hat{\eta} \geq 0, \qquad \forall C > 0, \tag{B.72}$$

and $\max\left\{\hat{m}^{c/C}, \hat{m}^{a/A}\right\} > 0$, where the definitions of $\hat{\varepsilon}^{Q,c/C}$ and $\hat{\varepsilon}^{Q,a/A}$ are given by (B.64). The last three results play a decisive role in the proofs of C) and D).

Proof of C)

Substituting (B.67) [= (B.31)] into (B.26) and taking into account (B.68) [= (B.32)] we obtain the instantaneous utility function (B.33)

$$u(c, c/C, a/A) = T\left(\xi_0 c^{\xi_1} Q(c/C, a/A)\right), \qquad \xi_0 > 0, \quad \xi_1 > 0. \tag{B.73}$$

Since according to (B.70), $m^{c/C}(C, 1, 1)$ is a constant function of C, the elasticity of $m^{c/C}$ with respect to own consumption c,

$$\varepsilon^{m^{c/C},c}(c, c/C, a/A) \equiv m_c^{c/C}(c, c/C, a/A) \times \left[m^{c/C}(c, c/C, a/A)\right]^{-1} c,$$

evaluated at $(c, c/C, a/A) = (C, 1, 1)$ has the property that

$$\varepsilon^{m^{c/C},c}(C, 1, 1) = 0, \qquad \forall C > 0.$$

Substituting the last result into the definition of the effective elasticity of intertemporal substitution given by (21),

$$\sigma^D(C) \equiv -\left[\varepsilon^{u_c,c}(C, 1, 1) + \frac{m^{c/C}(C, 1, 1)}{1 + m^{c/C}(C, 1, 1)} \times \varepsilon^{m^{c/C},c}(C, 1, 1)\right]^{-1},$$

we obtain

$$\sigma^D(C) = -\frac{1}{\varepsilon^{u_c,c}(C, 1, 1)}. \tag{B.74}$$

From (B.74) it follows that $\sigma^D(C)$ is a constant function of C if and only if $\varepsilon^{u_c,c}(C, 1, 1)$ is a constant function of C. From (B.50) and (B.53) it follows that the elasticity of the marginal utility of absolute consumption u_c with respect to absolute consumption c can be expressed in the following form:

$$\begin{aligned}
\varepsilon^{u_c,c}(c, c/C, a/A) &\equiv u_{cc}(c, c/C, a/A) \times [u_c(c, c/C, a/A)]^{-1} c \\
&= \varepsilon^{P,c}(c)\, \varepsilon^{T',v}(P(c)\, Q(c/C, a/A)) + \varepsilon^{P',c}(c),
\end{aligned}$$

where the elasticities $\varepsilon^{P,c}(c)$, $\varepsilon^{P',c}(c)$, and $\varepsilon^{T',v}(c)$ are defined in (B.54), (B.58), and (B.57). The expression $\varepsilon^{T',v}(P(c)\, Q(c/C, a/A))$ signifies that the elasticity $\varepsilon^{T',v}(v)$ is evaluated at $v = P(c)\, Q(c/C, a/A)$. The specification of $P(c)$ given by (B.67) [= (B.31)], $P(c) = \xi_0 c^{\xi_1}$, implies that $\varepsilon^{P,c}(c) = \xi_1$ and $\varepsilon^{P',c}(c) = \xi_1 - 1$ hold for $c > 0$. Using these results we obtain

$$\varepsilon^{u_c,c}(c, c/C, a/A) = \xi_1 \varepsilon^{T',v}\left(\xi_0 c^{\xi_1} Q(c/C, a/A)\right) + \xi_1 - 1.$$

In symmetric situations in which $(c, c/C, a/A) = (C, 1, 1)$ we thus have

$$\varepsilon^{u_c,c}(C, 1, 1) = \xi_1 \varepsilon^{T',v}\left(\xi_0 C^{\xi_1} Q(1, 1)\right) + \xi_1 - 1. \tag{B.75}$$

Obviously, $\varepsilon^{u_c,c}(C, 1, 1)$ and $\sigma^D(C) = -[\varepsilon^{u_c,c}(C, 1, 1)]^{-1}$ [see (B.74)] are constant functions of C if and only if $\varepsilon^{T',v}\left(\xi_0 C^{\xi_1} Q(1, 1)\right)$ is a constant function of C. Since $\xi_0 > 0$ and $\xi_1 > 0$ [see (B.68)] and $Q(c/C, a/A) > 0$ hold over the domain of Q [see (B.28)], $\varepsilon^{T',v}\left(\xi_0 C^{\xi_1} Q(1, 1)\right)$ is a constant function of C for $C > 0$ if and only if $\varepsilon^{T',v}(v)$ is a constant function of v for $v > 0$. The assumptions made in (B.27) require that $T' > 0$ and $T'' < 0$. Hence, admissible transformations T have the property that $\varepsilon^{T',v}(v) \equiv T''(v) \times [T'(v)]^{-1} v < 0$ holds for $v > 0$.

We can summarize these considerations as follows: If the transformation $T(v)$ is admissible in the sense that $T'(v) > 0$ and $T''(v) < 0$ hold for $v > 0$, then the instantaneous utility function (B.73), $u(c, c/C, a/A) = T\left(\xi_0 c^{\xi_1} Q(c/C, a/A)\right)$, has the property that $\varepsilon^{u_c,c}(C,1,1)$ and $\sigma^D(C) = -[\varepsilon^{u_c,c}(C,1,1)]^{-1}$ are constant functions of C for $C > 0$ if and only if the function T satisfies the condition

$$\varepsilon^{T',v}(v) = -\theta, \qquad v > 0, \tag{B.76}$$

where θ is an arbitrary strictly positive constant, $\theta > 0$. It is well-known that $\varepsilon^{T',v}(v) = -\theta < 0$ holds for $v > 0$ if and only if the function $T(v)$ is of the CRRA type, i.e.,

$$T(v) = \kappa_0 + \kappa_1 \frac{v^{1-\theta} - 1}{1-\theta}, \qquad v > 0, \tag{B.77}$$

where κ_0 and κ_1 are constants. These considerations prove the validity of the first assertion made in item C): The functions $\varepsilon^{u_c,c}(C,1,1)$ and $\sigma^D(C)$ that result from the specification of the instantaneous utility function $u(c, c/C, a/A) = T\left(\xi_0 c^{\xi_1} Q(c/C, a/A)\right)$ given by (B.33) [= (B.73)] have the properties described in (B.34),

$$\varepsilon^{u_c,c}(C,1,1) = \hat{\varepsilon}^{u_c,c}, \quad \sigma^D(C) = \hat{\sigma}, \qquad \forall C > 0,$$

if and only if the function $T(v)$ has the form given by (B.77) [= (B.35)].

Next, we derive the parameter restrictions given in (B.36). To ensure that $T' > 0$ and $T'' < 0$ hold as required by (B.27), we have to assume that $\kappa_1 > 0$ holds in addition to $\theta > 0$. Moreover, we have to ensure that either $P'' \leq 0$ holds or condition (B.29) satisfied.

The specification of $P(c)$ and $T(v)$ given by (B.67) [= (B.31)] and (B.77) [= (B.35)] imply that $P''(c) = -\xi_0 \xi_1 (1 - \xi_1) c^{\xi_1 - 2}$ and that condition (B.29) becomes $-\theta\xi_1 + \xi_1 - 1 < 0$. In the following we use the latter condition because it is weaker than the condition $\xi_1 \leq 1$ that results from the requirement that $P''(c) \leq 0$. We can summarize these considerations as follows: In order to ensure that $T'(v) > 0$ and $T''(v) < 0$ hold as required by (B.27), and that also the condition (B.29) is satisfied, it is assumed that

$$\kappa_1 > 0, \quad \theta > 0, \quad 1 + (\theta - 1)\xi_1 > 0 \tag{B.78}$$

holds in addition to (B.32). Obviously, the assumptions made in (B.78) are identical to those given in (B.36).

Using (B.75), (B.76), (B.78), and (B.74) we obtain

$$\varepsilon^{u_c,c}(C,1,1) = -[1 + \xi_1(\theta - 1)] \equiv \hat{\varepsilon}^{u_c,c} < 0, \qquad \forall C > 0, \tag{B.79}$$

$$\sigma^D(C) = -\frac{1}{\varepsilon^{u_c,c}(C,1,1)} = \frac{1}{1 + (\theta - 1)\xi_1} \equiv \hat{\sigma} > 0, \qquad \forall C > 0. \tag{B.80}$$

The last two results play an important role in the following proof of D).

Proof of D)

Substituting the specifications of $T(v)$ and $P(c)$ given by (B.77) [= (B.35)] and (B.67) [= (B.31)] into (B.26) and taking into account (B.60) [= (B.28)], (B.68) [= (B.32)] and (B.78) [= (B.36)], we obtain the instantaneous utility function

$$u(c, c/C, a/A) = \kappa_0 + \frac{\kappa_1}{1-\theta}\left\{\left[\xi_0 c^{\xi_1} Q(c/C, a/A)\right]^{1-\theta} - 1\right\}, \tag{B.81}$$

where the parameters satisfy the conditions

$$\kappa_1 > 0, \quad \xi_0 > 0, \quad \xi_1 > 0, \quad \theta > 0, \quad 1 + (\theta - 1)\xi_1 > 0 \tag{B.82}$$

and the function $Q(c/C, a/A)$ satisfies the conditions given in (B.28). The specification given by (B.81) is identical to the specification given by (B.37). The assumptions made in (B.82) are identical to the assumptions made in (B.38).

Proof of D-i) It is easily verified that

$$\begin{aligned}
u_c(c, c/C, a/A) &= \xi_1 \kappa_1 c^{-1}\left[\xi_0 c^{\xi_1} Q(c/C, a/A)\right]^{1-\theta} > 0,\\
u_{cc}(c, c/C, a/A) &= -\left[1 + \xi_1(\theta - 1)\right]\xi_1 \kappa_1 c^{-2}\left[\xi_0 c^{\xi_1} Q(c/C, a/A)\right]^{1-\theta} < 0,\\
u_{c/C}(c, c/C, a/A) &= \kappa_1 (c/C)^{-1}\left[\xi_0 c^{\xi_1} Q(c/C, a/A)\right]^{1-\theta} \varepsilon^{Q,c/C}(c/C, a/A) \geq 0,\\
u_{a/A}(c, c/C, a/A) &= \kappa_1 (a/A)^{-1}\left[\xi_0 c^{\xi_1} Q(c/C, a/A)\right]^{1-\theta} \varepsilon^{Q,a/A}(c/C, a/A) \geq 0.
\end{aligned}$$

The parameter restrictions given in (B.82) [= (B.38)] and the assumptions made in (B.60) [= (B.28)] with respect to the function $Q(c/C, a/A)$ ensure that the instantaneous utility function (B.81) [= (B.37)] is well-behaved in the sense that all assumptions made in (2) are satisfied, i.e.,

$$u_c > 0, \quad u_{cc} < 0, \quad u_{c/C} \geq 0, \quad u_{a/A} \geq 0, \quad u_{c/C} > 0 \vee u_{a/A} > 0.$$

Proof of D-ii) For the convenience of the reader, the tedious derivation of (B.39) and (B.40) given in D)-ii) is deferred to the end of the proof.

Proof of D-iii) From (B.70), (B.71), (B.64), and (B.79) it follows that the conditions given by (32) in Proposition 3 are satisfied, since $m^{c/C}(C,1,1) = \hat{m}^{c/C}$, $m^{a/A}(C,1,1) = \hat{m}^{a/A}$, and $\varepsilon^{u_c,c}(C,1,1) = \hat{\varepsilon}^{u_c,c}$ hold for $C > 0$, where

$$\hat{m}^{c/C} \equiv \frac{\hat{\varepsilon}^{Q,c/C}}{\xi_1} \geq 0, \quad \hat{m}^{a/A} \equiv \frac{\hat{\varepsilon}^{Q,a/A}}{\xi_1} \geq 0, \quad \hat{\varepsilon}^{u_c,c} \equiv -\left[1 + (\theta - 1)\xi_1\right] < 0. \tag{B.83}$$

The constants $\hat{\varepsilon}^{Q,c/C}$ and $\hat{\varepsilon}^{Q,a/A}$ denote the values that the elasticities of the function $Q(c/C, a/A)$ with respect to c/C and a/A, $\varepsilon^{Q,c/C}(c/C, a/A)$ and $\varepsilon^{Q,a/A}(c/C, a/A)$, take in symmetric situations, i.e., at $(c/C, a/A) = (1,1)$:

$$\hat{\varepsilon}^{Q,c/C} \equiv \varepsilon^{Q,c/C}(1,1), \quad \hat{\varepsilon}^{Q,a/A} \equiv \varepsilon^{Q,a/A}(1,1). \tag{B.84}$$

These results given in (B.83) and (B.84) prove the validity of (B.41) and (B.42).

Proof of D-iv) From (B.72), (B.64), and (B.80) it follows that the conditions given in (25) in Proposition 1 are satisfied, because $\sigma^D(C) = \hat{\sigma}$ and $\eta^D(C) = \hat{\eta}$ hold for $C > 0$, where

$$\hat{\sigma} \equiv \frac{1}{1 + (\theta - 1)\xi_1} > 0, \qquad \hat{\eta} \equiv \frac{\hat{\varepsilon}^{Q,a/A}/\xi_1}{1 + \hat{\varepsilon}^{Q,c/C}/\xi_1} \geq 0. \tag{B.85}$$

These results given in (B.85) prove the validity of (B.43).

We know from Proposition 1 that if these constants $\hat{\sigma}$ and $\hat{\eta}$ satisfy the condition (26),

$$\max\left\{\frac{[1 - (1/\hat{\sigma})][f_k(1, L) + \hat{\eta} f(1, L)]}{1 + \hat{\eta}}, 0\right\} < \rho < f_k(1, L) + \hat{\eta} f(1, L),$$

then an economically meaningful decentralized BGP exists. Substituting the results for $\hat{\sigma}$ and $\hat{\eta}$ given by (B.85) into equation (27) [see Proposition 1],

$$g^D = \frac{f_k(1, L) - \rho + \hat{\eta} f(1, L)}{(1/\hat{\sigma}) + \hat{\eta}},$$

we obtain

$$g^D = \frac{f_k(1, L) - \rho + \dfrac{\hat{\varepsilon}^{Q,a/A}/\xi_1}{1 + \hat{\varepsilon}^{Q,c/C}/\xi_1} \times f(1, L)}{1 + (\theta - 1)\xi_1 + \dfrac{\hat{\varepsilon}^{Q,a/A}/\xi_1}{1 + \hat{\varepsilon}^{Q,c/C}/\xi_1}}. \tag{B.86}$$

Obviously, Equation (B.86) is identical to Equation (B.44).

Proof of D-v) Since the validity of all assertions made in D-v) is verified at first glance, we skip the proof.

Proof of D-ii) Finally, we derive the conditions (B.39) and (B.40), which ensure that the alternative representation of the utility function given by $V(c, C, a, A) \equiv u(c, c/C, a/A)$ that results from (B.81) [= (B.37)], where the parameters and the function $Q(c/C, a/A)$ satisfy (B.38) and (B.28), respectively, is well-behaved, in the sense that all assumptions made in (4),

$$V_{cc} < 0, \qquad \text{and} \qquad V_{cc}V_{aa} - (V_{ca})^2 > 0 \text{ if } u_{a/A} > 0,$$

are satisfied. According to (A.1) and (A.2), we have

$$\begin{aligned} V_{cc} &= u_{cc} + 2C^{-1}u_{c(c/C)} + C^{-2}u_{(c/C)(c/C)} \\ V_{cc}V_{aa} - (V_{ca})^2 &= A^{-2}\left[u_{cc} + 2C^{-1}u_{c(c/C)} + C^{-2}u_{(c/C)(c/C)}\right]u_{(a/A)(a/A)} \\ &\quad -A^{-2}\left[u_{c(a/A)} + C^{-1}u_{(c/C)(a/A)}\right]^2. \end{aligned}$$

It can be shown that

$$\begin{aligned} u_c &= \xi_1\kappa_1\xi_0^{1-\theta}c^{\xi_1(1-\theta)-1}Q^{1-\theta}, \\ u_{cc} &= -[1 + \xi_1(\theta - 1)]\xi_1\kappa_1\xi_0^{1-\theta}c^{\xi_1(1-\theta)-2}Q^{1-\theta}, \\ u_{c/C} &= \kappa_1\xi_0^{1-\theta}c^{\xi_1(1-\theta)}Q^{-\theta}Q_{c/C} \\ &= \kappa_1(c/C)^{-1}\left(\xi_0c^{\xi_1}Q\right)^{1-\theta}\varepsilon^{Q,c/C}, \end{aligned}$$

$$
\begin{aligned}
u_{a/A} &= \kappa_1 \xi_0^{1-\theta} c^{\xi_1(1-\theta)} Q^{-\theta} Q_{a/A} \\
&= \kappa_1 (a/A)^{-1} \left(\xi_0 c^{\xi_1} Q\right)^{1-\theta} \varepsilon^{Q,a/A}, \\
u_{c(c/C)} &= (1-\theta)\, \xi_1 \kappa_1 \xi_0^{1-\theta} c^{\xi_1(1-\theta)-1} Q^{-\theta} Q_{c/C} \\
&= (1-\theta)\, \xi_1 \kappa_1 c^{-1} (c/C)^{-1} \left(\xi_0 c^{\xi_1} Q\right)^{1-\theta} \varepsilon^{Q,c/C}, \\
u_{c(a/A)} &= (1-\theta)\, \xi_1 \kappa_1 \xi_0^{1-\theta} c^{\xi_1(1-\theta)-1} Q^{-\theta} Q_{a/A} \\
&= (1-\theta)\, \xi_1 \kappa_1 c^{-1} (a/A)^{-1} \left(\xi_0 c^{\xi_1} Q\right)^{1-\theta} \varepsilon^{Q,a/A}, \\
u_{(c/C)(c/C)} &= \kappa_1 \xi_0^{1-\theta} c^{\xi_1(1-\theta)} \left[-\theta Q^{-\theta-1} Q_{c/C}^2 + Q^{-\theta} Q_{(c/C)(c/C)}\right] \\
&= \kappa_1 (c/C)^{-2} \left(\xi_0 c^{\xi_1} Q\right)^{1-\theta} \varepsilon^{Q,c/C} \left(-\theta \varepsilon^{Q,c/C} + \varepsilon^{Q_{c/C},c/C}\right), \\
u_{(c/C)(a/A)} &= \kappa_1 \xi_0^{1-\theta} c^{\xi_1(1-\theta)} \left(-\theta Q^{-\theta-1} Q_{a/A} Q_{c/C} + Q^{-\theta} Q_{(c/C)(a/A)}\right) \\
&= \kappa_1 (c/C)^{-1} (a/A)^{-1} \left(\xi_0 c^{\xi_1} Q\right)^{1-\theta} \varepsilon^{Q,c/C} \left(-\theta \varepsilon^{Q,a/A} + \varepsilon^{Q_{c/C},a/A}\right), \\
u_{(a/A)(a/A)} &= \kappa_1 \xi_0^{1-\theta} c^{\xi_1(1-\theta)} \left[-\theta Q^{-\theta-1} Q_{a/A}^2 + Q^{-\theta} Q_{(a/A)(a/A)}\right] \\
&= \kappa_1 (a/A)^{-2} \left(\xi_0 c^{\xi_1} Q\right)^{1-\theta} \varepsilon^{Q,a/A} \left(-\theta \varepsilon^{Q,a/A} + \varepsilon^{Q_{a/A},a/A}\right),
\end{aligned}
$$

where

$$
\begin{aligned}
\varepsilon^{Q,c/C}(c/C, a/A) &\equiv Q_{c/C}(c/C, a/A) \times \left[Q(c/C, a/A)\right]^{-1} (c/C), \\
\varepsilon^{Q,a/A}(c/C, a/A) &\equiv Q_{a/A}(c/C, a/A) \times \left[Q(c/C, a/A)\right]^{-1} (a/A), \\
\varepsilon^{Q_{c/C},c/C}(c/C, a/A) &\equiv Q_{(c/C)(c/C)}(c/C, a/A) \times \left[Q_{c/C}(c/C, a/A)\right]^{-1} (c/C), \\
\varepsilon^{Q_{c/C},a/A}(c/C, a/A) &\equiv Q_{(c/C)(a/A)}(c/C, a/A) \times \left[Q_{c/C}(c/C, a/A)\right]^{-1} (a/A), \\
\varepsilon^{Q_{a/A},a/A}(c/C, a/A) &\equiv Q_{(a/A)(a/A)}(c/C, a/A) \times \left[Q_{a/A}(c/C, a/A)\right]^{-1} (a/A)
\end{aligned}
$$

and

$$
Q > 0, \quad Q_{c/C} \geq 0, \quad Q_{a/A} \geq 0, \quad Q_{c/C} > 0 \vee Q_{a/A} > 0,
$$

$$
\kappa_1 > 0, \quad \xi_0 > 0, \quad \xi_1 > 0, \quad \theta > 0, \quad 1 + (\theta - 1)\, \xi_1 > 0,
$$

hold according to (B.28) and (B.38). Using these results it can be shown that

$$
\begin{aligned}
V_{cc} &= -\kappa_1 c^{-2} \left(\xi_0 c^{\xi_1} Q\right)^{1-\theta} \times \\
&\quad \times \left\{\left[1 + \xi_1 (\theta - 1)\right] \xi_1 + \varepsilon^{Q,c/C} \left[2(\theta - 1)\, \xi_1 + \theta \varepsilon^{Q,c/C} - \varepsilon^{Q_{c/C},c/C}\right]\right\}, \\
V_{aa} &= -\kappa_1 a^{-2} \left(\xi_0 c^{\xi_1} Q\right)^{1-\theta} \varepsilon^{Q,a/A} \left(\theta \varepsilon^{Q,a/A} - \varepsilon^{Q_{a/A},a/A}\right), \\
V_{ca} &= \kappa_1 (ca)^{-1} \left(\xi_0 c^{\xi_1} Q\right)^{1-\theta} \left\{\left[(1-\theta)\, \xi_1 - \theta \varepsilon^{Q,c/C}\right] \varepsilon^{Q,a/A} + \varepsilon^{Q,c/C} \varepsilon^{Q_{c/C},a/A}\right\}.
\end{aligned}
$$

Our assumptions imply that $V_{cc} < 0$ holds if and only if

$$
0 < \left[1 + \xi_1 (\theta - 1)\right] \xi_1 + \varepsilon^{Q,c/C} \left[2(\theta - 1)\, \xi_1 + \theta \varepsilon^{Q,c/C} - \varepsilon^{Q_{c/C},c/C}\right].
$$

This condition is identical to condition (B.39).

Moreover, $V_{cc}V_{aa} - (V_{ca})^2 > 0$ holds if and only if

$$\begin{aligned} 0 &< \left\{[1+\xi_1(\theta-1)]\,\xi_1 + \varepsilon^{Q,c/C}\left[2(\theta-1)\,\xi_1 + \theta\varepsilon^{Q,c/C} - \varepsilon^{Q_{c/C},c/C}\right]\right\} \times \\ &\qquad \times \varepsilon^{Q,a/A}\left(\theta\varepsilon^{Q,a/A} - \varepsilon^{Q_{a/A},a/A}\right) \\ &\quad - \left\{\left[(1-\theta)\,\xi_1 - \theta\varepsilon^{Q,c/C}\right]\varepsilon^{Q,a/A} + \varepsilon^{Q,c/C}\varepsilon^{Q_{c/C},a/A}\right\}^2. \end{aligned}$$

This second condition is only relevant if $\text{sgn}(u_{a/A}) = \text{sgn}(Q_{a/A}) > 0$. Obviously, it is identical to condition (B.40). ■

B.8 Proof of point A) of Corollary 1

Let the instantaneous utility function take the form given by (44), where the parameters satisfy the conditions given by (45) and (46):

$$u(c, c/C, a/A) = \frac{1}{1-\theta}\left\{\left[c^{\xi_1}(c/C)^{\xi_2}(a/A)^{\xi_3}\right]^{1-\theta} - 1\right\},$$

$$\theta > 0, \quad \xi_1 > 0, \quad \xi_2 \geq 0, \quad \xi_3 \geq 0, \quad \max\{\xi_2, \xi_3\} > 0,$$

$$(1-\theta)(\xi_1 + \xi_2 + \xi_3) < 1.$$

Obviously, the alternative representation of the utility function given by $V(c, C, a, A) \equiv u(c, c/C, a/A)$ takes the following form:

$$V(c, C, a, A) = \frac{1}{1-\theta}\left[\left(c^{\xi_1+\xi_2}C^{-\xi_2}a^{\xi_3}A^{-\xi_3}\right)^{1-\theta} - 1\right].$$

The following properties of u and V are easily verified:

$$\begin{aligned} u_c &= \xi_1 c^{-1}\left[c^{\xi_1}(c/C)^{\xi_2}(a/A)^{\xi_3}\right]^{1-\theta}, \\ u_{cc} &= -\xi_1[1+\xi_1(\theta-1)]\,c^{-2}\left[c^{\xi_1}(c/C)^{\xi_2}(a/A)^{\xi_3}\right]^{1-\theta}, \\ u_{c/C} &= \xi_2(c/C)^{-1}\left[c^{\xi_1}(c/C)^{\xi_2}(a/A)^{\xi_3}\right]^{1-\theta}, \\ u_{a/A} &= \xi_3(a/A)^{-1}\left[c^{\xi_1}(c/C)^{\xi_2}(a/A)^{\xi_3}\right]^{1-\theta}, \\ V_c &= (\xi_1+\xi_2)\,c^{-1}\left(c^{\xi_1+\xi_2}a^{\xi_3}C^{-\xi_2}A^{-\xi_3}\right)^{1-\theta}, \\ V_{cc} &= -(\xi_1+\xi_2)[1+(\xi_1+\xi_2)(\theta-1)]\,c^{-2}\left(c^{\xi_1+\xi_2}a^{\xi_3}C^{-\xi_2}A^{-\xi_3}\right)^{1-\theta}, \\ V_a &= \xi_3 a^{-1}\left(c^{\xi_1+\xi_2}a^{\xi_3}C^{-\xi_2}A^{-\xi_3}\right)^{1-\theta}, \\ V_{aa} &= -\xi_3[1+\xi_3(\theta-1)]\,a^{-2}\left(c^{\xi_1+\xi_2}a^{\xi_3}C^{-\xi_2}A^{-\xi_3}\right)^{1-\theta}, \\ V_{ca} &= (\xi_1+\xi_2)\,\xi_3(1-\theta)(ca)^{-1}\left(c^{\xi_1+\xi_2}a^{\xi_3}C^{-\xi_2}A^{-\xi_3}\right)^{1-\theta}, \end{aligned}$$

$$
\begin{aligned}
V_{cc}V_{aa} - (V_{ca})^2 &= (\xi_1 + \xi_2)\,\xi_3\,[1 + (\theta - 1)(\xi_1 + \xi_2 + \xi_3)] \times \\
&\quad \times (ca)^{-2}\left(c^{\xi_1+\xi_2} a^{\xi_3} C^{-\xi_2} A^{-\xi_3}\right)^{2(1-\theta)}.
\end{aligned}
$$

First, we prove that all assumptions made in (2) are satisfied. From the results given above it is obvious that

$$
\begin{aligned}
u_c &> 0 \Leftrightarrow \xi_1 > 0, \qquad u_{cc} < 0 \Leftrightarrow \xi_1\,[1 + \xi_1(\theta - 1)] > 0, \\
u_{c/C} &\geq 0 \Leftrightarrow \xi_2 \geq 0, \qquad u_{a/A} \geq 0 \Leftrightarrow \xi_3 \geq 0.
\end{aligned}
$$

Since, by assumption (45), $\xi_1 > 0$, $\xi_2 \geq 0$, $\xi_3 \geq 0$, and $\max\{\xi_2, \xi_3\} > 0$ hold, we obtain $u_c > 0$, $u_{c/C} \geq 0$, $u_{a/A} \geq 0$, and $u_{c/C} > 0 \vee u_{a/A} > 0$.

It remains to show that $u_{cc} < 0$ holds, too. Since $\xi_1 > 0$ holds by assumption we have

$$u_{cc} < 0 \Leftrightarrow (1 - \theta)\,\xi_1 < 1.$$

Obviously, $\theta \geq 1$ is sufficient (but not necessary) for $u_{cc} < 0$. In the opposite case in which $\theta < 1$ holds, we obtain

$$(1 - \theta)\,\xi_1 < (\xi_1 + \xi_2 + \xi_3)(1 - \theta),$$

where we made use of the fact that $\max\{\xi_2, \xi_3\} > 0$ holds due to assumption (45). Taking into account that $(1 - \theta)(\xi_1 + \xi_2 + \xi_3) < 1$ holds due to assumption (46), it is clear that $(1 - \theta)\,\xi_1 < 1$ so that the assumption $u_{cc} < 0$ is also satisfied if $\theta < 1$ holds.

Second, we prove that all assumptions made in (4) are satisfied. Since $\xi_1 > 0$ and $\xi_2 \geq 0$ hold by assumption, it follows from the expression for V_{cc} given above that

$$V_{cc} < 0 \Leftrightarrow (1 - \theta)(\xi_1 + \xi_2) < 1.$$

Obviously, $\theta \geq 1$ is sufficient (but not necessary) for $V_{cc} < 0$. In the opposite case in which $\theta < 1$ holds, we obtain

$$(1 - \theta)(\xi_1 + \xi_2) \leq (\xi_1 + \xi_2 + \xi_3)(1 - \theta).$$

Taking into account that $(1 - \theta)(\xi_1 + \xi_2 + \xi_3) < 1$ holds due to assumption (46) it is clear that $(1 - \theta)(\xi_1 + \xi_2) < 1$ so that the assumption $V_{cc} < 0$ is also satisfied if $\theta < 1$ holds.

Finally, we have to show that $V_{cc}V_{aa} - (V_{ca})^2 > 0$ holds if $u_{a/A} > 0$. Recall that $u_{a/A} > 0 \Leftrightarrow \xi_3 > 0$. If $\xi_3 > 0$ holds in addition to $\xi_1 > 0$ and $\xi_2 \geq 0$, then

$$V_{cc}V_{aa} - (V_{ca})^2 > 0 \Leftrightarrow (1 - \theta)(\xi_1 + \xi_2 + \xi_3) < 1.$$

Since the condition given on the right-hand side is satisfied due to assumption (46), we obtain $V_{cc}V_{aa} - (V_{ca})^2 > 0$ for $u_{a/A} > 0$. ■

C The decentralized solution – Part II (Section 4)

C.1 Generalization of (52)

Let u be of the (44)-type and exhibit the property that the parameter ξ_2 itself is a function of other parameters $\pi_1, \ldots, \pi_m$, $\xi_2 = \xi_2(\pi_1, \ldots, \pi_m)$, and the same is true for the parameters ξ_1, ξ_3, and θ. In this case the standard approach is confronted with the additional problem of identifying the parameter π_k out of $\{\pi_1, \ldots, \pi_m\}$ that seems to be the appropriate measure of the strength of the relative consumption motive. Its second problem is well-known from above: The derivative $dg^D/d\pi_k$ may deviate from the adequate measure $\partial g^D/\partial \hat{m}^{c/C}$ both quantitatively and qualitatively due to effects of changes in π_k via the $\hat{m}^{a/A}$- and the $|\hat{\varepsilon}^{u_c,c}|$-channels. In this more general case of functional dependence, the decomposition (52) has to be replaced by the following slightly more complicated equation:

$$\begin{aligned}
\frac{dg^D}{d\pi_k} &= \frac{\partial g^D}{\partial \hat{m}^{c/C}} \cdot \frac{d\hat{m}^{c/C}}{d\pi_k} + \frac{\partial g^D}{\partial \hat{m}^{a/A}} \cdot \frac{d\hat{m}^{a/A}}{d\pi_k} + \frac{\partial g^D}{\partial |\hat{\varepsilon}^{u_c,c}|} \cdot \frac{d|\hat{\varepsilon}^{u_c,c}|}{d\pi_k} \\
&= \frac{\partial g^D}{\partial \hat{m}^{c/C}} \left(\frac{1}{\xi_1} \frac{\partial \xi_2}{\partial \pi_k} - \frac{\xi_2}{\xi_1^2} \frac{\partial \xi_1}{\partial \pi_k} \right) + \frac{\partial g^D}{\partial \hat{m}^{a/A}} \left(\frac{1}{\xi_1} \frac{\partial \xi_3}{\partial \pi_k} - \frac{\xi_3}{\xi_1^2} \frac{\partial \xi_1}{\partial \pi_k} \right) \\
&\quad + \frac{\partial g^D}{\partial |\hat{\varepsilon}^{u_c,c}|} \left(\frac{\partial \theta}{\partial \pi_k} \xi_1 + (\theta - 1) \frac{\partial \xi_1}{\partial \pi_k} \right).
\end{aligned}$$

The validity of this decomposition is easily verified by using the fact that, according to (47) and (48) we have $\hat{m}^{c/C} = \xi_2/\xi_1$, $\hat{m}^{a/A} = \xi_3/\xi_1$, and $|\hat{\varepsilon}^{u_c,c}| = 1 + (\theta - 1)\xi_1$.

C.2 Appropriate restrictions on the Galí specification (55)

Let u take the Galí form given by (55)

$$u(c, c/C) = \frac{1}{1-\theta} \left[c^{[1-(1-\gamma)\theta]/(1-\theta)} (c/C)^{-\gamma\theta/(1-\theta)} \right]^{1-\theta}, \qquad \theta > 0. \tag{C.1}$$

It is obvious that $u = u(c, c/C)$ is of the type given by (44), where

$$\xi_1 = 1 - \xi_2 = \frac{1 - (1-\gamma)\theta}{1-\theta}, \qquad \xi_2 = -\frac{\gamma\theta}{1-\theta}, \qquad \xi_3 = 0, \tag{C.2}$$

while the irrelevant constant term "-1" is ignored.

Since $\xi_3 = 0$, the restrictions (45) and (46) given in Corollary 1 simplify to

$$\theta > 0, \quad \xi_1 > 0, \quad \xi_2 > 0, \quad (1-\theta)(\xi_1 + \xi_2) < 1. \tag{C.3}$$

If these restrictions are satisfied, then according to item A) of Corollary 1, the instantaneous utility function $u = u(c, c/C)$ given by (55) [= (C.1)] and the resulting representation of preferences given by $V(c, C, a, A) \equiv u(c, c/C, a/A)$ are well-behaved in the sense that all assumptions made in (2) and (4) are satisfied.

Taking into account that $\xi_1 + \xi_2 = 1$, it is obvious that the condition $(1-\theta)(\xi_1 + \xi_2) < 1$ given in (C.3) is satisfied. Hence, it remains to be shown that $\xi_1 > 0$ and $\xi_2 > 0$. It is clear that

$$\xi_2 > 0 \Leftrightarrow \text{sgn}\,(\gamma) = -\text{sgn}\,(1-\theta)\,, \tag{C.4}$$

$$\xi_1 > 0 \Leftrightarrow \text{sgn}\,(1-(1-\gamma)\,\theta) = \text{sgn}\,(1-\theta)\,. \tag{C.5}$$

The expression in (C.4) implies that we have to distinguish two cases with respect to θ.

Case A) Let $\theta > 1$. From (C.4) and (C.5) it follows that

$$\xi_2 > 0 \Leftrightarrow \gamma > 0, \tag{C.6}$$

$$\xi_1 > 0 \Leftrightarrow 1-(1-\gamma)\,\theta < 0 \Leftrightarrow \gamma < \frac{\theta-1}{\theta}. \tag{C.7}$$

Combining (C.6) and (C.7), we obtain

$$\theta > 1 \text{ and } 0 < \gamma < \frac{\theta-1}{\theta} < 1 \Rightarrow \xi_1 > 0, \quad \xi_2 > 0.$$

This completes the proof of case A) described in (56).

Case B) Let $\theta < 1$. From (C.4) and (C.5) it follows that

$$\xi_2 > 0 \Leftrightarrow \gamma < 0, \tag{C.8}$$

$$\xi_1 > 0 \Leftrightarrow 1-(1-\gamma)\,\theta > 0 \Leftrightarrow -\frac{1-\theta}{\theta} < \gamma. \tag{C.9}$$

Combining (C.8) and (C.9), we obtain

$$\theta < 1 \text{ and } -\frac{1-\theta}{\theta} < \gamma < 0 \Rightarrow \xi_1 > 0, \quad \xi_2 > 0.$$

This completes the proof of case B) described in (56).

C.3 Appropriate restrictions on the specification #3

In specification #3 the function $V = V\,(c, C)$ takes the form

$$V\,(c,C) = \frac{1}{1-\theta}\left\{\left[\left(\frac{c^{\varphi}-\kappa C^{\varphi}}{1-\kappa}\right)^{1/\varphi}\right]^{1-\theta} - 1\right\}, \qquad 0<\kappa<1, \quad 0<1-\varphi<\theta,$$

where the domain of V is given by $\Theta_V \equiv \{(c,C)\,|c>0, C>0, c^{\varphi}-\kappa C^{\varphi}>0\}$. The corresponding representation of the function $u = u\,(c, c/C)$ is given by (58),

$$u\,(c, c/C) = \frac{1}{1-\theta}\left\{\left[c\left(\frac{1-\kappa\,(c/C)^{-\varphi}}{1-\kappa}\right)^{1/\varphi}\right]^{1-\theta} - 1\right\}.$$

From

$$\begin{aligned}
V_c &= \left(\frac{c^{\varphi}-\kappa C^{\varphi}}{1-\kappa}\right)^{(1-\theta-\varphi)/\varphi}\frac{c^{\varphi-1}}{1-\kappa}, \\
V_{cc} &= -\left[\theta-(1-\varphi)\right]\left(\frac{c^{\varphi}-\kappa C^{\varphi}}{1-\kappa}\right)^{(1-\theta-2\varphi)/\varphi}\left(\frac{c^{\varphi-1}}{1-\kappa}\right)^{2} \\
&\quad -(1-\varphi)\left(\frac{c^{\varphi}-\kappa C^{\varphi}}{1-\kappa}\right)^{(1-\theta-\varphi)/\varphi}\frac{c^{\varphi-2}}{1-\kappa},
\end{aligned}$$

it follows that the assumptions $0<\kappa<1$ and $0<1-\varphi<\theta$ are sufficient for $V_c>0$ and $V_{cc}<0$ so that all assumptions made in (4) are satisfied.

From

$$\begin{aligned}
u_c &= c^{-\theta}\left(\frac{1-\kappa\left(c/C\right)^{-\varphi}}{1-\kappa}\right)^{(1-\theta)/\varphi}, \\
u_{cc} &= -\theta c^{-\theta-1}\left(\frac{1-\kappa\left(c/C\right)^{-\varphi}}{1-\kappa}\right)^{(1-\theta)/\varphi}, \\
u_{c/C} &= \frac{\kappa}{1-\kappa}c^{1-\theta}\left(\frac{1-\kappa\left(c/C\right)^{-\varphi}}{1-\kappa}\right)^{(1-\theta-\varphi)/\varphi}\left(c/C\right)^{-\varphi-1},
\end{aligned}$$

it follows that the assumptions $0<\kappa<1$ and $0<1-\varphi<\theta$ are also sufficient for $u_c>0$, $u_{cc}<0$, and $u_{c/C}>0$ so that all assumptions made in (2) are satisfied.

C.4 A generalized version of specification #6

We replace the simple specifications of $\tilde{u}(c,s)$ and $s(c/C,a/A)$ that are used in (64),

$$\tilde{u}(c,s)=\frac{1}{1-\theta}\left[\left(c^{1-\beta}s^{\beta}\right)^{1-\theta}-1\right], \qquad s(c/C,a/A)=(c/C)^{\phi}(a/A)^{1-\phi},$$

where $\theta>0$, $0<\beta<1$, $1+(\theta-1)(1-\beta)>0$, and $0<\phi<1$, by the more general functions

$$\tilde{u}(c,s)=\frac{1}{1-\theta}\left[\left(c^{\chi_1}s^{\chi_2}\right)^{1-\theta}-1\right], \quad \chi_1>0, \quad \chi_2>0, \quad 1+(\theta-1)\chi_1>0, \tag{C.10}$$

$$s(c/C,a/A)=(c/C)^{\phi_1}(a/A)^{\phi_2}, \quad \phi_1\geq 0, \quad \phi_2\geq 0, \quad \max\{\phi_1,\phi_2\}>0, \tag{C.11}$$

and do not impose any functional dependence on the five parameters θ, χ_1, χ_2, ϕ_1, and ϕ_2. The following properties of the utility function defined by (C.10) are easily verified:

$$m^s(c,s) \equiv \frac{s}{c}\times\frac{\tilde{u}_s(c,s)}{\tilde{u}_c(c,s)}=\frac{\chi_2}{\chi_1}\equiv\hat{m}^s>0, \quad \forall(c,s)\in\Theta_{\tilde{u}}, \tag{C.12}$$

$$\varepsilon^{\tilde{u}_c,c}(c,s) \equiv \frac{c\tilde{u}_{cc}(c,s)}{\tilde{u}_c(c,s)}=-\left[1+(\theta-1)\chi_1\right]\equiv\hat{\varepsilon}^{\tilde{u}_c,c}<0, \quad \forall(c,s)\in\Theta_{\tilde{u}}, \tag{C.13}$$

where $\Theta_{\tilde{u}}$ denotes the domain of the function $\tilde{u}(c,s)$, m^s represents the *percentage-MRS* of status s for absolute consumption c, and $\varepsilon^{\tilde{u}_c,c}$ denotes the elasticity of the marginal utility of absolute consumption $\tilde{u}_c(c,s)$ with respect to absolute consumption c. From (C.12)–(C.13) it is obvious that $m^s(c,s)$ and $\varepsilon^{\tilde{u}_c,c}(c,s)$ are constant functions over the domain $\Theta_{\tilde{u}}$. Consequently,

$\hat{m}^s$ measures the *global* strength of the quest for overall status (as determined by both relative consumption and relative wealth), while $1/\left|\hat{\varepsilon}^{\tilde{u}_c,c}\right|$ is a measure of the *global* willingness to substitute absolute consumption over time. The status function defined by (C.11) exhibits the property that its elasticities with respect to relative consumption and relative wealth are constant functions over the domain of $s\left(c/C, a/A\right)$ denoted by Θ_s, since

$$\varepsilon^{s,c/C}\left(c/C, a/A\right) = \phi_1 \equiv \hat{\varepsilon}^{s,c/C} \geq 0, \quad \varepsilon^{s,a/A}\left(c/C, a/A\right) = \phi_2 \equiv \hat{\varepsilon}^{s,a/A} \geq 0 \tag{C.14}$$

hold for all $\left(c/C, a/A\right) \in \Theta_s$.

The specifications given by (C.10) and (C.11) imply that the resulting representation of the instantaneous utility function $u\left(c, c/C, a/A\right) \equiv \tilde{u}\left(c, s\left(c/C, a/A\right)\right)$,

$$u\left(c, c/C, a/A\right) = \frac{1}{1-\theta}\left[\left(c^{\chi_1}\left(c/C\right)^{\chi_2\phi_1}\left(a/A\right)^{\chi_2\phi_2}\right)^{1-\theta} - 1\right], \tag{C.15}$$

is of the simple form given by (44) with

$$\xi_1 = \chi_1, \qquad \xi_2 = \chi_2\phi_1, \qquad \xi_3 = \chi_2\phi_2. \tag{C.16}$$

In contrast to Section 4 we do not restrict attention to the special case in which $\xi_1 + \xi_2 + \xi_3 = 1$ holds. Consequently, the resulting three fundamental factors are given by

$$\hat{m}^{c/C} = \frac{\chi_2}{\chi_1} \times \phi_1 \geq 0, \quad \hat{m}^{a/A} = \frac{\chi_2}{\chi_1} \times \phi_2 \geq 0, \quad \left|\hat{\varepsilon}^{u_c,c}\right| = \left[1 + \left(\theta - 1\right)\chi_1\right] > 0. \tag{C.17}$$

These results are either derived by substituting (C.16) into (47) and (48) or by using the fact that

$$\hat{m}^{c/C} = \hat{m}^s \times \hat{\varepsilon}^{s,c/C}, \qquad \hat{m}^{a/A} = \hat{m}^s \times \hat{\varepsilon}^{s,a/A}, \qquad \hat{\varepsilon}^{u_c,c} = \hat{\varepsilon}^{\tilde{u}_c,c}, \tag{C.18}$$

where $\hat{m}^s$, $\hat{\varepsilon}^{\tilde{u}_c,c}$, $\hat{\varepsilon}^{s,c/C}$, and $\hat{\varepsilon}^{s,a/A}$ are defined in (C.12)–(C.14). The results given in (C.17) and (C.18) imply that

$$\begin{aligned}
\hat{\sigma} &\equiv \frac{1}{\left|\hat{\varepsilon}^{u_c,c}\right|} = \frac{1}{\left|\hat{\varepsilon}^{\tilde{u}_c,c}\right|} = \frac{1}{1 + \left(\theta - 1\right)\chi_1}, \\
\hat{\eta} &\equiv \frac{\hat{m}^{a/A}}{1 + \hat{m}^{c/C}} = \frac{\hat{m}^s \times \hat{\varepsilon}^{s,a/A}}{1 + \hat{m}^s \times \hat{\varepsilon}^{s,c/C}} = \frac{\left(\chi_2/\chi_1\right) \times \phi_2}{1 + \left(\chi_2/\chi_1\right) \times \phi_1},
\end{aligned}$$

$$g^D = \frac{f_k\left(1, L\right) - \rho + \dfrac{\left(\chi_2/\chi_1\right) \times \phi_2}{1 + \left(\chi_2/\chi_1\right) \times \phi_1} f\left(1, L\right)}{1 + \left(\theta - 1\right)\chi_1 + \dfrac{\left(\chi_2/\chi_1\right) \times \phi_2}{1 + \left(\chi_2/\chi_1\right) \times \phi_1}}. \tag{C.19}$$

As long as the quite general specification (C.15) is used and no functional dependence between the five parameters θ, χ_1, χ_2, ϕ_1, and ϕ_2 is imposed, the application of the standard analysis allows for correct answers with respect to the implications of relative consumption and relative wealth preferences. The validity of this assertion is easily verified in three steps by using (C.17) and Proposition 4: 1) Changes in ϕ_1 affect only the percentage-MRS of relative consumption $\hat{m}^{c/C}$, while exerting no effect on $\hat{m}^{a/A}$ and $\left|\hat{\varepsilon}^{u_c,c}\right|$. Since $\hat{m}^{c/C}$ depends positively on ϕ_1, the qualitative effects of *ceteris paribus* changes in the strength of the relative consumption motive can be correctly inferred from the sign of $\partial g^D/\partial\phi_1$, where g^D is given by (C.19). 2) Changes in ϕ_2 affect only the percentage-MRS of relative wealth $\hat{m}^{a/A}$, while having no effect on $\hat{m}^{c/C}$ and $|\hat{\varepsilon}^{u_c,c}|$. Since $\hat{m}^{a/A}$ depends positively on ϕ_2, the sign of $\partial g^D/\partial\phi_2$ can be used to assess the qualitative implications of *ceteris paribus* changes in the strength of the relative wealth motive. 3) The following properties of $\partial g^D/\partial\phi_1$ and $\partial g^D/\partial\phi_2$ are easily verified: (a) $\partial g^D/\partial\phi_2 > 0$ holds regardless of whether $\phi_1 = 0$ or $\phi_1 > 0$. (b) If $\phi_2 = 0$, then g^D is independent of ϕ_1. (c) If $\phi_2 > 0$, then $\partial g^D/\partial\phi_1 < 0$. Hence, the standard analysis yields results that coincide with those given in Proposition 4: (a) The decentralized growth rate g^D depends positively on the strength of the relative wealth motive, irrespective of the strength of the relative consumption motive. (b) If relative wealth is irrelevant for utility, then relative consumption preferences do not affect g^D. (c) In the presence of relative wealth preferences, g^D depends negatively on the strength of the relative consumption motive.

Finally, the specification of the instantaneous utility function $\tilde{u}(c,s)$ given by (C.10) enables the standard approach to analyze changes in the strength of the quest for overall status that are not accompanied by concurrent changes in the willingness to substitute absolute consumption intertemporally. This possibility results from the fact that there is no functional dependence between the exponents of absolute consumption c and status s, χ_1 and χ_2. More precisely, a rise in χ_2 causes the percentage-MRS of status for absolute consumption $\hat{m}^s = \chi_2/\chi_1$ [see (C.12)] to increase, but leaves the elasticity of the marginal utility of absolute consumption $\tilde{u}_c(c,s)$ with respect to absolute consumption, $\hat{\varepsilon}^{\tilde{u}_c,c} = -[1+(\theta-1)\chi_1]$ [see (C.13)], unchanged. From $|\hat{\varepsilon}^{u_c,c}| = |\hat{\varepsilon}^{\tilde{u}_c,c}|$ [see (C.18)] it then follows that the fundamental factor $|\hat{\varepsilon}^{u_c,c}|$ and the resulting effective elasticity of intertemporal substitution $\hat{\sigma} = 1/|\hat{\varepsilon}^{u_c,c}|$ are also independent of the parameter χ_2. For convenience, in the rest of this analysis we restrict attention to the case in which both relative consumption and relative wealth matter for status so that $\phi_1 > 0$ and $\phi_2 > 0$. In this case the rise in $\hat{m}^s = \chi_2/\chi_1$ that results from the increase in χ_2 causes the fundamental factors $\hat{m}^{c/C}$ and $\hat{m}^{a/A}$ to rise. Please note that the percentage-MRS of relative wealth a/A for relative consumption c/C given by $m^{a/A}/m^{c/C} = \phi_2/\phi_1$ is unaffected by the increase in the strength of the overall status motive. The rise in $\hat{m}^{a/A}$ causes the comparison-induced extra return factor $\hat{\eta}$ to rise, while the rise in $\hat{m}^{c/C}$ leads to a decrease in $\hat{\eta}$. From $\partial\hat{\eta}/\partial\chi_2 = \chi_1\phi_2/(\chi_1+\chi_2\phi_1)^2$ it follows that the net effect is positive. Hence, the rise in χ_2 exerts a strictly positive effect on the decentralized growth rate g^D via the $\hat{\eta}$-channel. Since, as mentioned above, $|\hat{\varepsilon}^{u_c,c}|$ and, hence, $\hat{\sigma} = 1/|\hat{\varepsilon}^{u_c,c}|$ are independent of χ_2, there is no additional effect via the $\hat{\sigma}$-channel. These considerations show that the traditional analysis could in principle analyze the effects of a rise in the intensity of the quest for overall status that is not accompanied by a change in the willingness to substitute absolute consumption over time correctly by differentiating g^D given in (C.19) partially with respect to χ_2 and showing that if both relative consumption and relative wealth matter for status, i.e., $\phi_1 > 0$ and $\phi_2 > 0$ (recall that we restricted our attention to this case), then $\partial g^D/\partial\chi_2 > 0$.

D The socially planned solution and the inefficiency of the decentralized solution (Section 5)

D.1 Proof of Proposition 6

Proof of A) Let the instantaneous utility function u satisfy the conditions (32) that were introduced in the context of the decentralized economy in Proposition 3, i.e.,

$$m^{c/C}(C,1,1) = \hat{m}^{c/C}, \quad m^{a/A}(C,1,1) = \hat{m}^{a/A}, \quad \varepsilon^{u_c,c}(C,1,1) = \hat{\varepsilon}^{u_c,c}, \quad \forall C > 0, \tag{D.1}$$

where $\hat{m}^{c/C} \geq 0$, $\hat{m}^{a/A} \geq 0$ (with $\max\{\hat{m}^{c/C}, \hat{m}^{a/A}\} > 0$), and $\hat{\varepsilon}^{u_c,c} < 0$.

First, we prove the validity of (76). According to Proposition 3 the conditions given in (32) [= (D.1)] imply that

$$\sigma^D(C) = \frac{1}{|\hat{\varepsilon}^{u_c,c}|}, \qquad \forall C > 0. \tag{D.2}$$

Using the definition of $\sigma^P(C)$ given in (74),

$$\sigma^P(C) \equiv -\frac{1}{\varepsilon^{u_c,c}(C,1,1)},$$

and the condition

$$\varepsilon^{u_c,c}(C,1,1) = \hat{\varepsilon}^{u_c,c} < 0, \quad \forall C > 0$$

given in (32) [= (D.1)] we obtain

$$\sigma^P(C) = \frac{1}{|\hat{\varepsilon}^{u_c,c}|}, \qquad \forall C > 0. \tag{D.3}$$

Combining (D.2) and (D.3) we obtain (76):

$$\sigma^P(C) = \sigma^D(C) = \frac{1}{|\hat{\varepsilon}^{u_c,c}|} \equiv \hat{\sigma}, \qquad \forall C > 0.$$

Second, we derive the solutions for g^P and $(C/K)^P$. Substitution of $\sigma^P(C) = \hat{\sigma}$, $\forall C > 0$, into the Euler equation of aggregate consumption in the socially planned economy that is given by (74), we obtain

$$\dot{C}/C = \hat{\sigma}\left[f(1,L) - \rho\right]. \tag{D.4}$$

From the economy's resource constraint $\dot{K} = f(1,L)K - C$ it follows that

$$\dot{K}/K = f(1,L) - (C/K). \tag{D.5}$$

Taking into account that, by assumption, L is exogenously given and constant over time and that $\hat{\sigma}$ is a constant, it is obvious from the last two differential equations that there exists a balanced growth path (BGP) in the socially planned economy in which C and K grow at the same constant rate so that C/K remains unchanged over time. The steady-state value of the common growth rate of aggregate consumption and aggregate physical capital denoted by $g^P = (\dot{C}/C)^P = (\dot{K}/K)^P$ and the steady-state value of the consumption-capital ratio denoted by $(C/K)^P$ are determined by the following system of equations:

$$\begin{aligned} g^P &= \hat{\sigma}\left[f(1,L)-\rho\right], \\ g^P &= f(1,L)-(C/K)^P. \end{aligned}$$

Solving this system of two equations for g^P and $(C/K)^P$, we obtain

$$g^P = \hat{\sigma}\left[f(1,L)-\rho\right], \tag{D.6}$$

$$(C/K)^P = (1-\hat{\sigma})\, f(1,L) + \hat{\sigma}\rho. \tag{D.7}$$

Third, we derive condition (77) and show that this condition implies the validity of (78). Using (D.6), we obtain

$$g^P > 0 \Leftrightarrow \rho < f(1,L). \tag{D.8}$$

From (D.7) it follows that

$$(C/K)^P > 0 \Leftrightarrow \rho > (\hat{\sigma}-1)(\hat{\sigma})^{-1} f(1,L). \tag{D.9}$$

In case that $\hat{\sigma} < 1$, condition (D.9) is redundant because $\rho > 0$ holds by assumption.

Along the BGP we have $\dot{K}/K = g^P$ at any point in time. Hence, the transversality condition (75),

$$\lim_{t\to\infty} e^{-f(1,L)t} K(t) = 0,$$

requires that

$$-f(1,L) + g^P = -\left[(1-\hat{\sigma})\, f(1,L) + \hat{\sigma}\rho\right] = -(C/K)^P < 0. \tag{D.10}$$

Obviously, the condition that $\rho > (\hat{\sigma}-1)(\hat{\sigma})^{-1} f(1,L)$ given in (D.9) implies not only that $(C/K)^P > 0$, but also ensures that the transversality condition is satisfied.

The results given by (D.8), (D.9), and (D.10) can be summarized as follows: If the condition

$$\max\left\{\frac{(\hat{\sigma}-1)\, f(1,L)}{\hat{\sigma}}, 0\right\} < \rho < f(1,L) \tag{D.11}$$

is satisfied, then the BGP is economically meaningful in the sense that the growth rate and the consumption-capital ratio are strictly positive,

$$g^P = \hat{\sigma}\left[f(1,L)-\rho\right] > 0, \qquad (C/K)^P = (1-\hat{\sigma})\, f(1,L) + \hat{\sigma}\rho > 0, \tag{D.12}$$

and, in addition, the transversality condition is fulfilled. Obviously, condition (D.11) is identical to condition (77). Moreover, (D.12) is identical to (78). ■

Proof of B) Finally, we show that if the condition (32) [= (D.1)] is satisfied, then the model has no transitional dynamics. Let $Z \equiv C/K$. Since K is a state variable and C is a control variable, $Z = C/K$ is a control-like variable (this notion is used by Barro and Sala-i-Martin (1995) on p. 162). In contrast to K, both C and $Z = C/K$ can jump at a certain point in time.

Using (D.4), (D.5), and $C/K = Z$, we obtain the following differential equation:

$$\begin{aligned}\dot{Z} &= [(\dot{C}/C) - (\dot{K}/K)]Z \\ &= \{\hat{\sigma}\,[f\,(1,L) - \rho] - [f\,(1,L) - Z]\}\,Z \\ &= \{Z - [(1-\hat{\sigma})\,f\,(1,L) + \hat{\sigma}\rho]\}\,Z \equiv \Phi\,(Z)\,.\end{aligned}$$

Solving $\dot{Z} = \Phi\,(Z) = 0$ for Z, we obtain $\{Z = 0\}$ and $\{Z = Z^P\}$, where

$$Z^P = (1-\hat{\sigma})\,f\,(1,L) + \hat{\sigma}\rho. \tag{D.13}$$

Obviously, Z^P given by (D.13) is identical to $(C/K)^P$ given by (D.7). If (77) [= (D.11)] holds, then $Z^P = (C/K)^P > 0$, so that Z^P is the economically meaningful steady-state value of the consumption-capital ratio. Rewriting $\Phi\,(Z)$ as $\Phi\,(Z) = \left(Z - Z^P\right)Z$ it is easily verified that

$$\Phi'\,(Z) = Z + \left(Z - Z^P\right), \qquad \Phi'\left(Z^P\right) = Z^P > 0.$$

$\Phi'\left(Z^P\right) > 0$ implies that the economically meaningful steady state of the differential equation $\dot{Z} = \Phi\,(Z)$ is unstable. Hence, the perfect-foresight equilibrium path of Z has no transitional dynamics, i.e., $Z\,(t) = Z^P$ for $t \geq 0$. The initial value of the jump variable Z has to be chosen in such a way that $Z\,(0) = Z^P$. From $Z = C/K$ and $Z^P = (C/K)^P$ it then follows that the initial value of the jump variable C has to be chosen according to $C\,(0) = (C/K)^P \times K_0$, where $(C/K)^P$ is given by (78) [= (D.7)] and K_0 is exogenously given.

From $Z\,(t) = Z^P$ for $t \geq 0$, (D.4), (D.5), $Z = C/K$, and (D.12) it then follows that

$$\begin{aligned}\dot{C}/C &= \hat{\sigma}\,[f\,(1,L) - \rho] = g^P > 0, \\ \dot{K}/K &= f\,(1,L) - Z^P = \hat{\sigma}\,[f\,(1,L) - \rho] = g^P > 0,\end{aligned}$$

hold for $t \geq 0$. The growth rates of consumption and capital are constant over time, identical, and equal to g^P. Consequently, the growth rates of C and K have no transitional dynamics. ■

D.2 Proof of Proposition 7

Proof of i) We assume that the conditions given by (26) and (77),

$$\max\left\{\frac{[1-(1/\hat{\sigma})]\,[f_k\,(1,L) + \hat{\eta}f\,(1,L)]}{1+\hat{\eta}}, 0\right\} < \rho < f_k\,(1,L) + \hat{\eta}f\,(1,L)\,,$$

$$\max\left\{\frac{(\hat{\sigma}-1)\,f\,(1,L)}{\hat{\sigma}}, 0\right\} < \rho < f\,(1,L)\,,$$

are satisfied so that in both the decentralized economy and the socially planned economy an economically meaningful BGP exists. The corresponding solutions for g^P and g^D are given by [see (78) and (27)]:

$$g^P = \hat{\sigma}\left[f(1,L)-\rho\right] > 0, \tag{D.14}$$
$$g^D = \frac{f_k(1,L)-\rho+\hat{\eta}f(1,L)}{(1/\hat{\sigma})+\hat{\eta}} > 0. \tag{D.15}$$

From (D.14) it is obvious that the socially optimal growth rate g^P is independent of $\hat{\eta}$:

$$\frac{\partial g^P}{\partial \hat{\eta}} = 0. \tag{D.16}$$

According to (B.25) the decentralized growth rate g^D depends positively on $\hat{\eta}$:

$$\frac{\partial g^D}{\partial \hat{\eta}} = \frac{(1/\hat{\sigma})f(1,L)-\left[f_k(1,L)-\rho\right]}{\left[(1/\hat{\sigma})+\hat{\eta}\right]^2} = \frac{(C/K)^D}{(1/\hat{\sigma})+\hat{\eta}} > 0.$$

Combining the last two results we obtain

$$\frac{\partial\left(g^P-g^D\right)}{\partial \hat{\eta}} = -\frac{\partial g^D}{\partial \hat{\eta}} < 0. \tag{D.17}$$

It is easily verified that the growth rate gap g^P-g^D can be written in the following two forms:

$$\begin{aligned} g^P-g^D &= \frac{\left[f(1,L)-f_k(1,L)\right]-\left[(1-\hat{\sigma})f(1,L)+\hat{\sigma}\rho\right]\hat{\eta}}{(1/\hat{\sigma})+\hat{\eta}} \\ &= \frac{(1-\hat{\sigma})f(1,L)+\hat{\sigma}\rho}{(1/\hat{\sigma})+\hat{\eta}} \times \left(\frac{f(1,L)-f_k(1,L)}{(1-\hat{\sigma})f(1,L)+\hat{\sigma}\rho}-\hat{\eta}\right). \end{aligned}$$

From the second line it follows that g^P-g^D can be expressed as

$$g^P-g^D = \Lambda \times \left(\hat{\eta}^{\text{crit}}-\hat{\eta}\right), \tag{D.18}$$

where

$$\Lambda \equiv \frac{(1-\hat{\sigma})f(1,L)+\hat{\sigma}\rho}{(1/\hat{\sigma})+\hat{\eta}}, \qquad \hat{\eta}^{\text{crit}} \equiv \frac{f(1,L)-f_k(1,L)}{(1-\hat{\sigma})f(1,L)+\hat{\sigma}\rho}. \tag{D.19}$$

Condition (77) implies that $(1-\hat{\sigma})f(1,L)+\hat{\sigma}\rho > 0$. From (24) it follows that $f(1,L) > f_k(1,L)$. Hence, both $\Lambda > 0$ and $\hat{\eta}^{\text{crit}}$ are strictly positive:

$$\Lambda > 0, \qquad \hat{\eta}^{\text{crit}} > 0. \tag{D.20}$$

From (D.18) and (D.20) it follows that

$$\operatorname{sgn}\left(g^P-g^D\right) = \operatorname{sgn}\left(\hat{\eta}^{\text{crit}}-\hat{\eta}\right). \tag{D.21}$$

Using (D.16), (D.17), and (D.19)–(D.21) we obtain the following summary of mathematical assertions that is identical to the compilation given in Proposition 7 at the end of item i):

$$\frac{\partial g^P}{\partial \hat{\eta}} = 0, \quad \frac{\partial \left(g^P - g^D\right)}{\partial \hat{\eta}} = -\frac{\partial g^D}{\partial \hat{\eta}} < 0,$$

$$\operatorname{sgn}\left(g^P - g^D\right) = \operatorname{sgn}\left(\hat{\eta}^{\text{crit}} - \hat{\eta}\right), \quad \text{where } \hat{\eta}^{\text{crit}} \equiv \frac{f(1,L) - f_k(1,L)}{(1-\hat{\sigma}) f(1,L) + \hat{\sigma}\rho} > 0.$$

The validity of the interpretation of these mathematical results that is given at the beginning of item i) is obvious. ■

Proof of ii) From (D.14) it follows that g^P depends positively on $\hat{\sigma}$:

$$\frac{\partial g^P}{\partial \hat{\sigma}} = [f(1,L) - \rho] = \frac{1}{\hat{\sigma}} g^P > 0.$$

According to (B.24) g^D depends positively on $\hat{\sigma}$, too:

$$\frac{\partial g^D}{\partial \hat{\sigma}} = \frac{f_k(1,L) - \rho + \hat{\eta} f(1,L)}{\hat{\sigma}^2 \left[(1/\hat{\sigma}) + \hat{\eta}\right]^2} = \frac{g^D}{\hat{\sigma}^2 \left[(1/\hat{\sigma}) + \hat{\eta}\right]} > 0.$$

The last two results imply that

$$\begin{aligned} \frac{\partial \left(g^P - g^D\right)}{\partial \hat{\sigma}} &= \frac{1}{\hat{\sigma}} \left(g^P - \frac{g^D}{1 + \hat{\sigma}\hat{\eta}} \right) \\ &= \frac{1}{\hat{\sigma}} \left(g^P - g^D\right) + \frac{\hat{\eta}}{1 + \hat{\sigma}\hat{\eta}} g^D \geq \frac{1}{\hat{\sigma}} \left(g^P - g^D\right), \end{aligned}$$

These results yield the following compilation of mathematical assertions that is given in Proposition 7 at the end of item ii):

$$\frac{\partial g^P}{\partial \hat{\sigma}} > 0, \quad \frac{\partial g^D}{\partial \hat{\sigma}} > 0, \quad \frac{\partial \left(g^P - g^D\right)}{\partial \hat{\sigma}} = \frac{1}{\hat{\sigma}} \left(g^P - \frac{g^D}{1 + \hat{\sigma}\hat{\eta}} \right) \geq \frac{1}{\hat{\sigma}} \left(g^P - g^D\right).$$

The validity of the interpretation of these mathematical results that is given at the beginning of item ii) is verified at first glance. ■

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University of Hohenheim
Dean’s Office of the Faculty of Business, Economics and Social Sciences
Palace Hohenheim 1 B
70593 Stuttgart | Germany
Fon +49 (0)711 459 22488
Fax +49 (0)711 459 22785
wiso@uni-hohenheim.de
wiso.uni-hohenheim.de